Optimal control theory 2

We did not cover the conditions properly last lecture, which was spent on examples.

This lecture will start at

- "Necessity": In this course[note], you can take as necessary for (x^*, u^*) to solve the problem, that:

  There exists a $p = p(t)$, continuous and piecewise C', s.t. the max. principle holds.

- Sufficiency: [see previous note]:
  - → Mangasarian: H concave in (x, u)
  - → Arrow: $\hat{H}(t, x, p) = \max_{u \in U} H(t, x, u, p)$ is concave in x.

& . . next page ff.

[note:] That means: put the "p_0" constant $= 1$. Note that p_0 is <u>not</u> $p(0)$! (The "$p_0 = 0$" cases is, kind of, a "constraint qualification fails" thing.)

Sensitivity / shadow prices. Let V = optimal value.

Fact: We have, to the level of precision this course requires, that:

$$\frac{\partial V}{\partial x_0}(x_0, x_1, t_0, t_1) = p(t_0)$$

$$\frac{\partial V}{\partial x_1}(x_0, x_1, t_0, t_1) = -p(t_1).$$

[x_1 is a "constraint". Reducing x_1 slacks an $x(t_1) \geq x_1$ constraint.]

$$\frac{\partial V}{\partial t_0}(x_0, x_1, t_0, t_1) = -H(t_0, x_0, u^*(t_0), p(t_0))$$

[increasing t_0 decreases the timespan!]

$$\frac{\partial V}{\partial t_1}(x_0, x_1, t_0, t_1) = H(t_1, x^*(t_1), u^*(t_1), p(t_1))$$

[increasing the timespan]

though: derivatives may be discontinuous.

Ex: $\max \int_0^T e^{-\delta t} u \, dt$ s.t. $\dot{x} = -u$, $x(T) \geq 0$; $u \in [0,1]$

We had $p(t) \equiv \bar{p}$, and if $x_0 > T$ then $\bar{p} = 0$;

If $x_0 < T$ then $\bar{p} = e^{-\delta T}$; if $x_0 = T$, any $\bar{p} \in [0, e^{-\delta T}]$

At $x_0 = T$, discontinuous derivative:

- Increasing $x_0 - T$: keep the $u^* \equiv 1$ solution. Derivatives as case $x_0 > T$.
- Decreasing $x_0 - T$: the $u^*(t) = \begin{cases} 1 & \text{if } t < x_0 \\ 0 & \text{otherwise} \end{cases}$ solution. Derivatives as case $x_0 < T$.

Example: last lecture's example 3: $\frac{\partial V}{\partial T} = ?$

Problem was $\max_{u \text{ 0 or } -1} \int_0^T (-x^2 - u^2)\, dt$, $\dot{x} = u \in \{-1, 0\}$

$x(t_0) > 0$, $x(T)$ free.

$H(t, x, u, p) = -x^2 - u^2 + pu$

$u^* = 0$ if $p > -1$, and because $x(T)$ free,

we have $p(T) = 0$.

So:

$$\frac{\partial V}{\partial T} = H(T, x^*(T), u^*(T), p(T)) = -\left(x^*(T)\right)^2$$

(We could calculate that: $x^*(T) = x^*(t^*) = x_0 - t^*$

where $2(x_0 - t^*)(T - t^*) = 1$. $x_0 - t^* = \frac{1}{2(T - t^*)}$

where $(x_0 - T + T - t^*)(T - t^*) = \frac{1}{2}$

$$(T - t^*)^2 + (x_0 - T)(T - t^*) - \tfrac{1}{2} = 0$$

$$2(T - t^*) = -(x_0 - T) + \sqrt{(x_0 - T)^2 + 2}$$

$$x^*(T) = \frac{1}{2(T - t^*)} = \frac{1}{-(\;) + \sqrt{\;}} \cdot \frac{x_0 - T + \sqrt{\;}}{x_0 - T + \sqrt{\;}}$$

$$= \frac{1}{2}\left(x_0 - T + \sqrt{(x_0 - T)^2 + 2}\right)$$

$$\frac{\partial V}{\partial T} = -\left(x^*(T)\right)^2 = -\frac{1}{2}\left((x_0 - T)^2 + (x_0 - T) + 1\right)$$

The "current value" formulation.

You can do without this on the exam, but it could be very useful; in particular, when $f(t,x,u) = e^{-\delta t}\underbrace{\tilde{f}(x,u)}_{\text{no "}t\text{"}}$ and $\underbrace{\frac{\partial g}{\partial t} \equiv 0}_{\text{no "}t\text{"}}$.

* Here is how it goes: if $f = e^{-\delta t}\tilde{f}$,

$$H(t,x,u,p) = \underbrace{e^{-\delta t}}_{\text{positive}}\Big[\underbrace{\tilde{f} + e^{\delta t} p\, g}_{u^* \text{ maximizes this.}}\Big]$$

Put $\lambda = e^{\delta t} p$. Then

$$\dot\lambda = \delta e^{\delta t} p + e^{\delta t}\dot p$$
$$= \delta\lambda - \frac{\partial}{\partial x}\left[\tilde{f} + \lambda g\right]$$

Transversality cond's for λ? same as for p.

* What do we gain?
  - → If neither g nor the current-value running utility $\tilde{f}$ have explicit t-dependence, then the system $\binom{\dot x}{\dot\lambda}$ has no explicit t-dependence. [If it is linear, we can solve it!]
  - → Interpretation: $\lambda(t)$ = the value – current at t – of one more unit of x.

Note: Arrow & Mangasarian: "same" for $H^{cv} = \tilde{f} + \lambda g$ as for H !

Example (from the book, slightly tweaked)

$$\max_{u \geq 0} \int_0^T (4K - u^2) e^{-0.25t} dt, \quad \dot{K} = u - 0.25K$$

$$K(0) = \underset{>0}{k_0}, \quad K(T) \text{ free}$$

(but automatically >0!)

Current-value Hamiltonian

$$H^{cv}(K, u, \lambda) = 4K - u^2 + \lambda(u - 0.25K)$$

Conditions: $u^* = \begin{cases} 0 & \text{if } \lambda < 0 \\ \lambda/2 & \text{if } \lambda \geq 0 \end{cases}$

$$\dot{\lambda} - 0.25\lambda = -\frac{\partial}{\partial K} H^{cv} = -4 + 0.25\lambda, \quad \lambda(T) = 0$$

and $\dot{K} = -0.25K + \frac{1}{2}\max\{0, \lambda\}$

Mangasarian or Arrow do apply, so we can guess that $\lambda \geq 0$, check that we get a candidate satisfying the maximum principle, and if so, conclude. The system is then linear, and can be solved explicitly: $\dot{\lambda} - \frac{1}{2}\lambda = -4$, $\lambda(T) = 0$

$\Rightarrow \lambda = 8 \cdot (1 - e^{-\frac{1}{2}(T-t)})$

and $\dot{K} + \frac{1}{4}K = 4(1 - e^{-\frac{1}{2}(T-t)})$ ....

Why should that be true? Answer as an economist!

[calculate!]

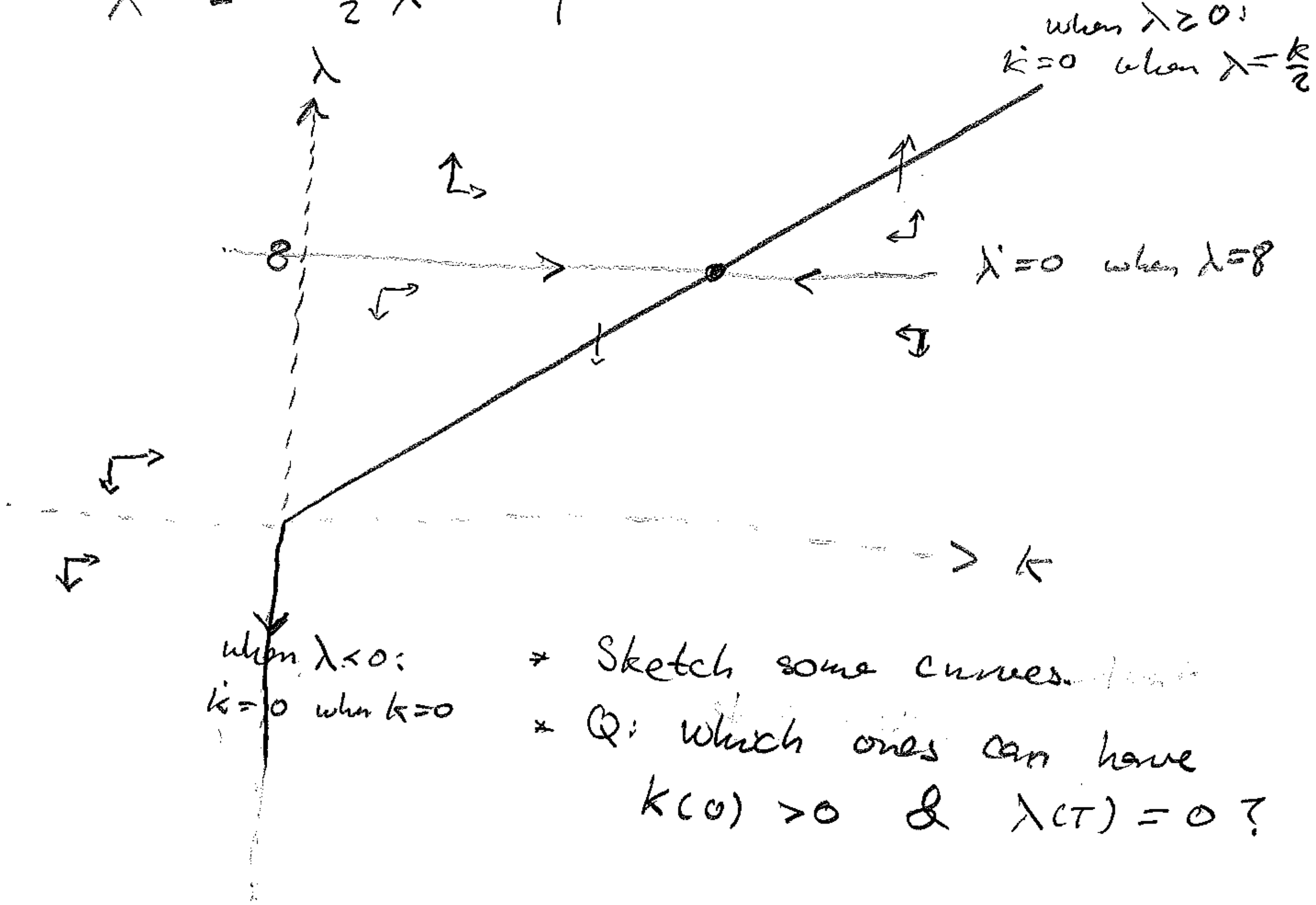
But we can also get a lot of qualitative information from a phase diagram

(this since the system is autonomous!)

means: no explicit "t"

$$\dot{k} = -\tfrac{1}{4}k + \tfrac{1}{2}\max\{0, \lambda\}$$

$$\dot{\lambda} = \tfrac{1}{2}\lambda - 4$$



* Sketch some curves.
* Q: Which ones can have $k(0) > 0$ & $\lambda(T) = 0$?

Further questions:

- Fix $k_0 > 0$. If T increases, how would you guess the initial $\lambda(0)$ must change? What happens as $T \to +\infty$?

- Classify the stationary state $(16, 8)$