Optimal control theory, lecture 3

We are essentially done with curriculum. This lecture will cover more examples, starting on the phase diagram for the last problem on the previous note.

Curriculum for phase diagrams:

→ Phase diagrams for diff. eq. systems [w/o optimization] are curriculum.

→ These could be helpful in optimal control too – e.g. that problem mentioned above.

→ Infinite horizon is not curriculum, so FMEA 9.12 / MA2 12.12 is only relevant "to the extent relevant for other purposes" like the two previous items.

"Graphical tool ex" 1 : yesterday's example. ← Hide

"Graphical tool ex" 2 : The below, except $\int_0^T$ and $K(T) \geq 0$. Question: where could we/ can we not end up at T?

EXAMPLE 2 Consider an economy with capital stock $K = K(t)$ and production per unit of time $Y = Y(t)$, where $Y = aK - bK^2$, with a and b as positive constants. Consumption is $C > 0$, whereas $Y - C = aK - bK^2 - C$ is investment. Over the period $[0, \infty)$, the objective is to maximize total discounted utility. In particular we consider the problem

$$\int_0^\infty \frac{1}{1-v} C^{1-v} e^{-rt} dt, \qquad \dot{K} = aK - bK^2 - C, \qquad K(0) = K_0 > 0$$

where $a > r > 0$ and $v > 0$, with C as the control variable. We require that

$$K(t) \geq 0 \text{ for all } t$$

The current value Hamiltonian is $H^c = \frac{1}{1-v} C^{1-v} + \lambda(aK - bK^2 - C)$. An interior maximum of H^c requires $\partial H^c / \partial c = 0$, i.e.

$$C^{-v} = \lambda \tag{i}$$

The differential equation for $\lambda = \lambda(t)$ is $\dot{\lambda} = -\lambda(a - 2bK) + r\lambda$, or

$$\dot{\lambda} = \lambda(r - a + 2bK) = 2b\lambda\left(K - \frac{a-r}{2b}\right) \tag{ii}$$

Now (i) implies that $C = \lambda^{-1/v}$, which inserted into the differential equation for K yields

$$\dot{K} = aK - bK^2 - \lambda^{-1/v} \tag{iii}$$

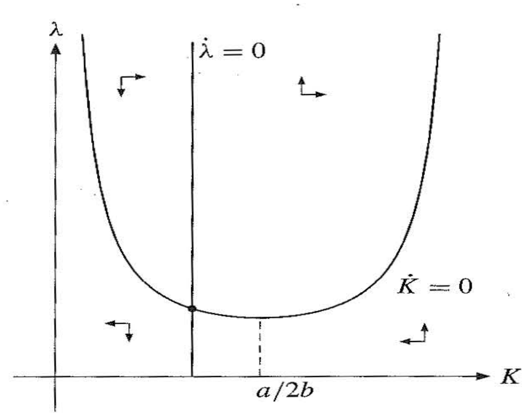

Figure 2

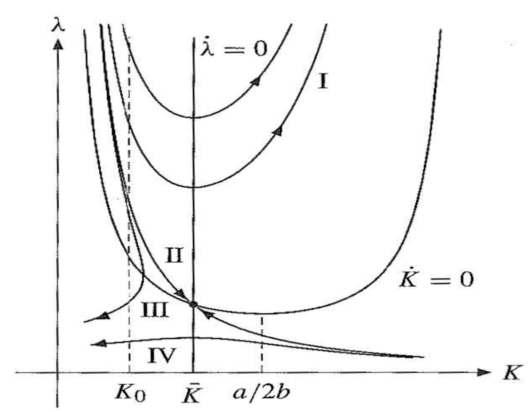

Figure 3

Figure 2 presents a phase diagram for the system given by (ii) and (iii). We see that $\dot{K} = 0$ for $\lambda = (aK - bK^2)^{-v}$, with $v > 0$. Here $z = aK - bK^2$ represents a concave parabola with $z = 0$ for $K = 0$ and for $K = a/b$. For $z = 0$, one has $\lambda = \infty$. The graph of $\dot{K} = 0$ is symmetrical about $K = a/2b$. Note that $\dot{\lambda} = 0$ when $K = (a - r)/2b$, which gives a straight line parallel to the λ-axis. Because $0 < (a - r)/2b < a/2b$, the graph of $\dot{\lambda} = 0$ will be as suggested in the figure. The equilibrium point $(\bar{K}, \bar{\lambda})$ is given by $\bar{K} = (a - r)/2b$, $\bar{\lambda} = \left[(a^2 - r^2)/4b\right]^{-v}$.

More examples

Ex FMEA ex 9.1.2/9.8.1. (Extraction of non-renewable.)

Fix a function $q(t) > 0$, and a $C(t,u)$ increasing, strictly convex in u.

$$\max_{u \geq 0} \int_0^T [q(t)u(t) - C(t,u(t))] e^{-rt}\,dt \quad \text{s.t. } \dot{x} = -u$$
$$x(0) = x_0 > 0, \qquad x(T) \geq 0.$$

T fixed, as usual.

H, current-value: $(q - \lambda)u - C(t,u)$

u^* maximizes: $\frac{\partial C}{\partial u} = q - \lambda$ defines maximum, by strict convexity

λ satisfies $\dot{\lambda} = r\lambda + 0$ with $\lambda(T) \ldots\ldots$

so $\lambda = Ae^{rt}$ $(\lambda(T) > 0 \Rightarrow x(T) = 0)$

So, using FOC u:

$$\underbrace{q - \frac{\partial C}{\partial u}}_{\text{marginal mining profit}} = \overbrace{Ae^{rt}}^{\lambda} \quad \text{(Hotelling!)}$$

grows at discount rate

Given that $\int_0^T u(t)\,dt = x_0$ (as $x(T) = 0$),

this determines A and thus optimal path.

What if T free to choose? Then $H = 0$ at T:

$$u^*(T)q(T) - C(T,u^*(T)) = \lambda u^*(T) = u^*(T)Ae^{rT}$$

$$\frac{u^*(T)q(T) - C(T,u^*(T))}{u^*(T)} = \lambda = Ae^{rT} = q(T) - \frac{\partial C}{\partial u}(T,u^*(T))$$

Simplify to: $\frac{\partial C}{\partial u}(T,u^*(T)) = \frac{C(T,u^*(T))}{u^*(T)}$

which says ..... ?

Even more examples, if time permits:

Curriculum note "Example 1" (do the "rescaled" p 8)

Compendium 9-15.

→ also: if final time = T and initial state is $x_0 > 0$,

→ how small must T be so that x stays <u>finite</u> for all u?

→ For what T and x_0 is it optimal to use $u^* = 0$?