Constrained max/min: precise necessary cond's

Problem: $\max f(\vec{x})$ s.t $\begin{cases} g_1(\vec{x}) & \begin{cases} \le b_1 \text{ or} \\ = b_1 \end{cases} \\ \vdots \\ g_m(\vec{x}) & \begin{cases} \le b_m \text{ or} \\ = b_m \end{cases} \end{cases}$ (P)

if equality-constraints only: "min" is OK too.

Terminology:

- **Admissible** point: one that satisfies the constraints (admissibility is part of the Lagrange cond's)
- **Active** constraint at $\vec{x}^*$: if $g_j(\vec{x}^*) = b_j$

  notes:
  - Lagrange problem: all constraints active.
  - possible that $g_j(\vec{x}^*) = b_j$ and $\lambda_j = 0$. !
- Lagrangian: $L(\vec{x}) = f(\vec{x}) - \sum_{j=1}^{m} \lambda_j \cdot (g_j(\vec{x}) - b_j)$

The "Math 2" conditions:

$$\left.\begin{array}{l} * \quad \vec{x}^* \text{ admissible} \\ * \quad \nabla L(\vec{x}^*) = \vec{0} \\ * \quad \text{For ineq. constraints:} \\ \qquad \lambda_j \ge 0 \ (= 0 \text{ if inactive constraint}) \end{array}\right] \quad (*)$$

Q: when are conditions (*) necessary?

Standing assumption (until we start on the specifics of concave programming):

All functions are C^1.

(But think: why do the Lagrange cond's for $\max \underbrace{x^{1/2} y^{1/4}}_{\text{Cobb-Douglas}}$ s.t $px + y = m$ produce only the max pt, not the min?]

Necessary conditions: For an admissible $\vec{x}^*$ to solve problem (P), either there exist unique numbers $\lambda_1, \dots, \lambda_m$ such that (*) holds, or:

- Eq-constraints only problem: the row vectors $\nabla g_1(\vec{x}^*), \dots, \nabla g_m(\vec{x}^*)$ are linearly dependent
- Ineq/eq constraints: delete inactive constraints, the remaining $\{\nabla g_j(\vec{x}^*)\}$ linearly dependent.

Alternative formulations: consider the negation of the latter, i.e:

(CQ) [the gradients of the active constraints' g_j form a linearly independent set

This is called a constraint qualification (CQ) ("føringsbetingelse" in NO)

* Necessary is then: (*) holds or (CQ) fails.

* The so-called "Fritz John" cond's unify these: in place of $\nabla L(\vec{x}^*) = \vec{0}$, write

$$\lambda_0 \nabla f(\vec{x}^*) = \sum_{j=1}^{m} \lambda_j \nabla g_j(\vec{x}^*),$$

and: $\lambda_0 = 0$, some $\lambda_j \neq 0$ ($\Rightarrow$ CQ fails)

or: $\lambda_0 = 1$ ($\Rightarrow \nabla L = 0$) and $\vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ unique.

Note: if all the constraints are linear then "CQ fails" only affects uniqueness of $\vec{\lambda}$; drop the uniqueness cond's, and (*) will produce all possible solution points.

When can the CQ fail? Examples

→ Inconveniently **formulated** problem:

Ex 1: As $g(x)=b \iff (g(x)-b)^3=0$, try the latter:

$$\nabla f = \hat\lambda \nabla[(g-b)^3]$$
$$= 3\hat\lambda \underbrace{(g(x^*)-b)^2}_{=0} \nabla g(x^*) = 0.$$

A "good" specification could (?) yield $\nabla f = \lambda \nabla g$ [note]

Ex 2: Repeated constraint!

[Only uniqueness of $\hat\lambda$ is affected]

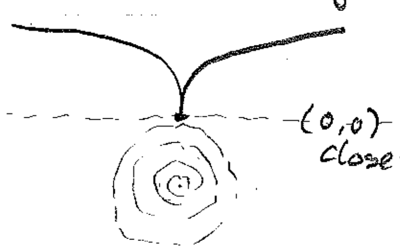
has been given on an exam!

→ Admissible set ends in a "tip" (when the tangent turns 180° around)

Ex 1: Ineq constraints $x \geq 0$ $y \geq 0$ $y \leq x^2$

(0,0) "must always be checked, no matter f!"

Ex 2: $\min\ x^2+(y+1)^2$ s.t. $y^3 = |x|^{3/2}$



Lagrange conds:

$$2x = \begin{cases} 0 & \text{if } x=0 \\ \frac{3}{2}\lambda\sqrt{|x|}\,\operatorname{sign} x & \text{otherwise} \end{cases}$$

$2(y+1) = 3\lambda y^2$ → fails to produce $y=0$!

Re [note] above: try to get a "good" specification, say $y = \sqrt{|x|}$ – but then C^1 fails!

→ Admissible set is (/has) an isolated point! Stupid problem, eh? Well there could be parameters involved, and at some values the admissible set collapses.

Examples: Recall the mean – variance trade-off problems. Make a twist: assume a risk-free opportunity does not exist; Implement that by imposing $\underbrace{\vec{1}^T \vec{x}}_{\text{total risky amount}} = W$ (=wealth) so that total risky amount = total endowment.

First the model for risk-averse agents:

$$\min \vec{x}^T \underset{\text{pos.def}}{\vec{A}} \vec{x} \quad \text{subject to} \quad \underbrace{\vec{\eta}^T \vec{x} = d, \quad \vec{1}^T \vec{x} = W}_{\text{linear.}}$$

(CQ) fails if $\vec{\eta} = \alpha \vec{1}$ for some α, in which case d must be $= \alpha W$ and the first constraint can be dropped. But the Lagrangian will still be stationary

$$2 \vec{x}^T \vec{A} = \lambda \vec{\eta}^T + \varkappa \vec{1}^T \quad \text{(whether or not CQ holds)}$$

[This is also two-fund separation: $\vec{A}^{-1}\vec{\eta}$, $\vec{A}^{-1}\vec{1}$ – and no safe! Cf "zero-beta CAPM"]

From $\vec{x} = \frac{\lambda}{2} \vec{A}^{-1} \vec{\eta} + \frac{\varkappa}{2} \vec{A}^{-1} \vec{1}$ and constraints

$$W = \vec{1}^T \vec{x} = \frac{\lambda}{2} \vec{1}^T \vec{A}^{-1} \vec{\eta} + \frac{\varkappa}{2} \vec{1}^T \vec{A}^{-1} \vec{1}$$

$$d = \vec{\eta}^T \vec{x} = \frac{\lambda}{2} \vec{\eta}^T \vec{A}^{-1} \vec{\eta} + \frac{\varkappa}{2} \vec{\eta}^T \vec{A}^{-1} \vec{1}$$

Two eq's for $\binom{\varkappa}{\lambda}$. CQ holds $\Rightarrow$ unique $\binom{\varkappa}{\lambda}$

If CQ fails: $\vec{x} = \frac{\lambda + \alpha \varkappa}{2} \vec{A}^{-1} \vec{1}$

$$W = \frac{\lambda + \alpha\varkappa}{2} \vec{1}^T \vec{A}^{-1} \vec{1}$$

"Degree of freedom" $\varkappa \Leftrightarrow \lambda$, but $\vec{x}$ will be unique!

What about the model that does <u>not</u> assume risk aversion?

$$\max \vec{\eta}^T \vec{x} \quad \text{s.t} \quad \vec{x}^T \vec{A} \vec{x} = Q^2 \ , \quad \vec{1}^T \vec{x} = W$$

If CQ holds, $\vec{\eta}^T = \Lambda \cdot 2\vec{x}^T \vec{A} + K \vec{1}^T$

capital λ, κ

Either $\Lambda \neq 0$, leads to $\vec{x} = \frac{1}{2\Lambda}\left(\vec{A}^{-1}\vec{\eta} - K\vec{A}^{-1}\vec{1}\right)$

and calculations akin to the risk-averse case

or $\Lambda = 0$, possible only iff $\vec{\eta}$ is a scalar of $\vec{1}$. If $\vec{\eta} = \alpha \vec{1}$, $K = \alpha$, $\vec{\eta}^T \vec{1} = \alpha W$, but a non-risk averse agent need not choose the minimal Q^2.

But any "fund" $\perp \vec{1}$ — pure risk, no return — will do.

When can CQ fail?

When $\Lambda\, 2\vec{x}^T \vec{A} + K \vec{1}^T = 0$ and not both $\Lambda = 0$, $K = 0$.

both cases: Q^2 chosen so that the admissible set is a singleton; yields $\vec{x} = \frac{W}{\vec{1}^T \vec{A}^{-1} \vec{1}} \vec{A}^{-1} \vec{1}$, the "minimum variance portfolio".

→ $\vec{x} = \vec{0}$. Possible for zero-wealth agents. Yes – in this model we must [if $\vec{\eta} \neq \alpha \vec{1}$] check the CQ to allow a zero-wealth agent to choose $\vec{x} = \vec{0}$!

→ When the hyperplane $\vec{1}^T \vec{x} = W$ is tangent to the ellipsoid $\vec{x}^T \vec{A} \vec{x} = Q^2$.

What is exam relevant concerning the CQ?

- Know that Lagrange / KT isn't the whole truth
  - there is something called constraint qualification
- Know that a "tip" when the boundary of the admissible set turns $180°$, must be checked separately.

Why the CQ? (optional, but "exercise")

The rest of this page is not essential, but a nice exercise in differentiating implicit functions: Consider the eq. system ($m+1 \leq n$ eq's)

$$g_1(\vec{x}) = b_1, \ldots, g_m(\vec{x}) = b_m, \quad f(\vec{x}) = \varphi$$

- Can these $m+1$ eq's define $m+1$ of the x_i as function of everything else, locally? If so, then
  - → we are not at a max/min, because we can increase/decrease φ slightly
  - → $\underbrace{\begin{pmatrix} \nabla g_1 \\ \vdots \\ \nabla g_m \\ \nabla f \end{pmatrix}}$ $d\vec{x} = \begin{pmatrix} d\vec{b} \\ d\varphi \end{pmatrix}$

    this has full rank. And as there are $m+1 \leq n$ rows and n col's, it means lin. indep rows:

    $0 = \lambda_1 \nabla g_1(\vec{x}^*) + \ldots + \lambda_m \nabla g_m(\vec{x}^*) + \lambda_{m+1} \nabla f(\vec{x}^*)$

    not all the $\lambda_i = 0$. Put $\lambda_0 = -\lambda_{m+1}$.

So for max/min, we must have linear dependence. If the $\{\nabla g_i\}$ lin. dep $\Rightarrow$ ——— " ——— (CQ fails). Otherwise, introducing $\nabla f(\vec{x}^*)$ must cause lin. dep $\Rightarrow$ Lagrange cond's.