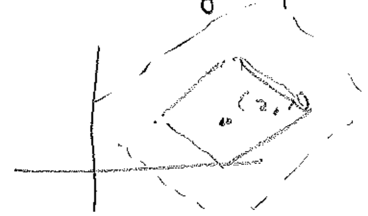
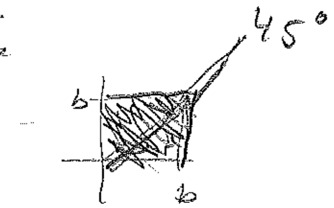
Example: $f = -|x-2| - |y-1|$ concave.

Level curves:



prefer: close to $(2,1)$

$\max f(x,y)$ s.t. $\underbrace{\max\{x,y\}}_{\text{convex}} \leq b$



b constant $\in [\frac{1}{2}, \frac{3}{2}]$.

In particular: as $\max\{x,y\} \leq \frac{3}{2}$,

$-|2-x| = \underline{x-2}$.

$$L(x,y) = x-2-|y-1| - \lambda\left(\max\{x,y\} - b\right)$$

Derivatives:

$$\frac{\partial L}{\partial x} = \begin{cases} 1 & \text{if } x<y \\ 1-\lambda & \text{if } x>y \end{cases}$$

so we <u>cannot</u> have $x<y$; for then, we can increase x w/o violating the constraint.

$$\frac{\partial L}{\partial y} = \begin{cases} 1 & \text{if } y<1 \ \& \ y<x \\ 1-\lambda & \text{if } x<y<1 \quad \text{(impossible that } x<y\text{!)} \\ -1 & \text{if } x>y>1 \\ -1-\lambda & \text{if } y>1 \ \& \ y>x \quad \text{(impossible that } y>x\text{)} \end{cases}$$

Note: not differentiable everywhere.

Can very well have $x=y$.

First: Not only is $x \geq y$; we also have $x=b$.

(Otherwise, $\lambda=0$ and we just increase x.)

So we are down to:

Either $x = y = b$ or $y < x = b$.

* Case $x = y = b$.

Only possible if $b \le 1$: If $b > 1$, we would have $\frac{\partial L}{\partial y} < 0$ and we can improve by decreasing y.

If $b \le 1$, $L(x, y) = x + y - 3 - \lambda \left(\max\{x, y\} - b \right)$

Any $\lambda \ge 2$ will make $(x, y) = (b, b)$ maximize.

(Why not "$\lambda = 1$"? Move (x, y) in direction $(1,1)$

$$L(x+h, y+h) = L(x, y) + 2h - \begin{cases} \lambda h & \text{if } x \neq y \\ 2\lambda h & \text{if } x = y \end{cases}$$
)

So: $(x^*, y^*) = (b, b)$ ok for $b \le 1$.

* Case $y < x = b$. Only possible for $b \ge 1$.

(Otherwise, $\frac{\partial L}{\partial y} = 1$ so increase y.)

If $x > y$, $L(x, y) = x \cdot (1 - \lambda) - |y - 1| - 2 + \lambda b$

Must have $\lambda = 1 = y$, and that works.

Example

max $2\sqrt{x^3 y^5}$ s.t $x + y \le b$

$$L(x,y) = 2x^{3/2} y^{5/2} - \lambda(x+y-b)$$

Conditions:

$$3x^{1/2} y^{5/2} = \lambda$$

$$5x^{3/2} y^{3/2} = \lambda$$

$$\lambda \ge 0 \quad (=0 \text{ if } x+y<b)$$

Conditions satisfied – with $\lambda = 0$ – wherever $xy = 0$.

Assume $\lambda > 0$ (so $x+y=b$)

Then $\frac{3}{5} x^{-1} y = 1$

$$y = \frac{5}{3} x,$$

So $x = \frac{3}{8} b$ and $y = \frac{5}{8} b$

* Obviously, this point is the solution.

* <u>Question</u>: How to prove that without rewriting the problem and without using the extreme value theorem?