Eigenvalues, square roots, ...

Knowing the conclusion of this page, is not curriculum.

Being able to understand the calculations, <u>is</u> curriculum. Take it as exercise.

Let $\vec{V}^{n\times n}$ have eigenvectors of $\vec{A}^{n\times n}$ as columns.

Then $\vec{A}\vec{V} = \left(\vec{A}\vec{v}^{(1)} \,\middle|\, \cdots \,\middle|\, \vec{A}\vec{v}^{(n)}\right)$

$$= \vec{V}\vec{\Lambda} \quad \text{where } \vec{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

A symmetric matrix has n lin. indep eigenvectors, Stacking these into $\vec{V}$, it is invertible.

$$\vec{A} = \vec{V}\vec{\Lambda}\vec{V}^{-1}$$

High powers? $\vec{A}^{2018} = \vec{V}\vec{\Lambda}^{2018}\vec{V}^{-1}$ ☺

Low powers, like ... $\frac{1}{2}$? $\vec{V}\begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}\vec{V}^{-1}$?

Ok if $\vec{A}$. pos. semidef or pos. def.

Fact: a pos. semidef $\vec{A}$ has a unique <u>pos. semidef square root</u> $\vec{S}$. often denoted $\vec{A}^{1/2}$.

If $\vec{Y}$ = random $\sim E\vec{Y} = \vec{0}$, $E\vec{Y}\vec{Y}^T = \vec{A}$ invertible, what is the covar of $\vec{Z} = (\vec{A}^{-1})^{1/2}\vec{Y}$? ...

Quadratic forms under linear constraints

Consider a restriction to a subspace: $\vec{B}\vec{x} = \vec{0}$

($\vec{B}$ can be assumed to have full rank).

$Q = \vec{x}^T \vec{A} \vec{x}$ is called pos. def. subject to $\vec{B}\vec{x} = \vec{0}$
if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$ such that $\vec{B}\vec{x} = \vec{0}$.

Pos semidef subject to $\vec{B}\vec{x} = \vec{0}$ if ...

Neg ——— " ——— ...

Neg. def ——— " ——— ...

Indef ——— " ——— ...

Fact: $\vec{A}$ pos. def $\Rightarrow$ pos. def subject to $\vec{B}\vec{x} = \vec{0}$

but not $\Leftarrow$

neg neg

But Q indefinite subject to $\vec{B}\vec{x} = \vec{0}$ $\Rightarrow$ $\vec{A}$ indef.

Application: approximate a Lagrange problem

Criteria ...? Definiteness only, not semidef.

First, in order for the following to work, the <u>leftmost</u> $m \times m$ minor of the full-rank $m \times n$ matrix $\vec{B}$ must be nonzero.

If not: re-enumerate the variables, redefine $\vec{A}$ and $\vec{B}$.

Form the bordered Hessian

$$\begin{pmatrix} \vec{O}_{m \times m} & \vec{B} \\ \vec{B}^T & \vec{A} \end{pmatrix}$$

Let b_r = the leading principal $(m+r) \times (m+r)$ minor – covering "down to variable r"

Then:

Q pos. def. subject to $\vec{B}\vec{x} = \vec{0}$

$$\Updownarrow \qquad \downarrow ?$$

$(-1)^m b_r > 0$ for all $r = m+1, \dots, n$

Q neg. def subject to $\vec{B}\vec{x} = \vec{0}$

$$\Updownarrow$$

$(-1)^r b_r > 0$ for all $r \geq m+1$.

$\vec{A}$ indef.

Example:
$$\vec{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad \vec{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -9 & 1 \end{pmatrix}$$

Let's be lazy. First row of $\vec{B}$ says $x_1 = 0$.
Just put $x_1 = 0$ to get the following in $\vec{y} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix}$:

$$\vec{A}_{-1} \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{B}_{-1} = \begin{pmatrix} \underbrace{\begin{matrix} 1 & 1 \\ 0 & -9 \end{matrix}}_{\text{nonzero minor}} & \begin{matrix} 0 \\ 1 \end{matrix} \end{pmatrix}$$

Form
$$\begin{pmatrix} 0 & 0 & \vec{B} \\ 0 & 0 & \\ \vec{B}^T & & \vec{A} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -9 & 1 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & -9 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

I shall never check $\lrcorner$, remember?

2 constraints, b_1 is 3×3, shall check
b_{m+1} and up, b_{m+1} is 5×5.

Cofactor exp.: $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -9 \\ 1 & 0 & 1 & 2 \\ 1 & -9 & 2 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -9 & \\ 1 & 1 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{vmatrix}$

$$= -81 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 81 - 4$$

$m = 2$ now, so $(-1)^m b_r > 0$, $r = 3 \ldots 3$. pos. def s.t. the constraint