

Eigenvalues, square roots, ...

Knowing the conclusion of this page, is not curriculum.

Being able to understand the calculations, is curriculum. Take it as exercise.

Let  $\vec{V}$  have eigenvectors of  $\vec{A}$  as columns.

$$\begin{aligned} \text{Then } \vec{A} \vec{V} &= \left( \vec{A} \vec{v}^{(1)} \mid \dots \mid \vec{A} \vec{v}^{(n)} \right) \\ &= \vec{V} \vec{\Lambda} \text{ where } \vec{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \end{aligned}$$

A symmetric matrix has  $n$  lin. indep eigenvectors, stacking these into  $\vec{V}$ , it is invertible.

$$\vec{A} = \vec{V} \vec{\Lambda} \vec{V}^{-1}$$

High powers?  $\vec{A}^{2018} = \vec{V} \vec{\Lambda}^{2018} \vec{V}^{-1}$  ☺

Low powers, like ...  $\frac{1}{2}$ ?  $\vec{V} \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix} \vec{V}^{-1}$  ?

Ok if  $\vec{A}$  pos. semidef or pos. def.

Fact: a pos. semidef  $\vec{A}$  has a unique pos. semidef square root  $\vec{S}$  often denoted  $\vec{A}^{1/2}$ .

If  $\vec{Y} = \text{random } n \times 1 \text{ } E \vec{Y} = \vec{0}$ ,  $E \vec{Y} \vec{Y}^T = \vec{A}$  invertible,

what is the covar of  $\vec{Z} = (\vec{A}^{-1})^{1/2} \vec{Y}$  ?



First, in order for the following to work, the leftmost  $m \times m$  minor of the full-rank  $m \times n$  matrix  $\vec{B}$  must be nonzero.

If not: re-enumerate the variables, redefine  $\vec{A}$  and  $\vec{B}$ .

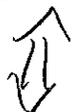
Form the bordered Hessian

$$\begin{pmatrix} \vec{0}_{m \times m} & \vec{B} \\ \vec{B}^T & \vec{A} \end{pmatrix}$$

Let  $b_r$  = the leading principal  $(m+r) \times (m+r)$  minor - covering "down" to variable  $r$

Then:

$\mathcal{Q}$  pos. def. subject to  $\vec{B}_x \vec{x} = \vec{0}$



$(-1)^m b_r > 0$  for all  $r = m+1, \dots, n$



$\mathcal{Q}$  neg. def. subject to  $\vec{B}_x \vec{x} = \vec{0}$



$(-1)^r b_r > 0$  for all  $r \geq m+1$ .

Example:

$\vec{A}$  indef.

$$\vec{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -9 & 1 \end{pmatrix}$$

Let's be lazy. First row of  $\vec{B}$  says  $x_1 = 0$ .

Just put  $x_1 = 0$  to get the following in  $\vec{y} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix}$ :

$$\vec{A}_{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \vec{B}_{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -9 & 1 \end{pmatrix}$$

nonzero minor

Form

$$\begin{pmatrix} 0 & 0 & \vec{B} \\ \vec{B}^T & \vec{A} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -9 & 1 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & -9 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

I shall never check  $\_$ , remember?

2 constraints,  $b_2$  is  $3 \times 3$ , shall check

$b_{m+1}$  and up,  $b_{m+1}$  is  $5 \times 5$ .

Cofactor exp:  $\begin{vmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -9 & 1 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & -9 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -9 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & -9 & -1 & 0 \end{vmatrix}$

$$= -81 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 81 - 4$$

$m = 2$  now, so  $(-1)^m b_r > 0, r = 3 \dots 3$ .

pos. def  
s.t the  
constraint