

Convex sets

(no: we do not speak about "concave" sets)

Recall: • convex combination of \vec{u} and \vec{v}
= $\lambda \vec{u} + (1-\lambda) \vec{v}$ where $\lambda \in [0,1]$

• convex combination of $\vec{v}^{(1)}, \dots, \vec{v}^{(k)}$
= $\lambda_1 \vec{v}^{(1)} + \dots + \lambda_k \vec{v}^{(k)}$, each $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$.

Def: Let S be a subset of some vector space, say \mathbb{R}^n
(so that lin. comb's are well-defined!)

S is convex if, whenever $\vec{u} \in S$ and $\vec{v} \in S$,
we also have $\lambda \vec{u} + (1-\lambda) \vec{v} \in S$, any $\lambda \in (0,1)$.

Also: strictly convex if $\lambda \vec{u} + (1-\lambda) \vec{v}$ is in addition
never a boundary point unless $\vec{u} = \vec{v}$ and \vec{v} on boundary.
(if $\lambda \in (0,1)$)

Ex: • \mathbb{R}^n itself!

And \emptyset and any singleton - but no other finite set!

• In \mathbb{R}^2 : the intervals! Indeed, "convex set" is
a generalization of "interval". And; book uses $[\vec{u}, \vec{v}]$
for $\{\vec{w}; \text{conv. comb of } \vec{u}, \vec{v}\}$.

• Given $\vec{a}, r: \{\vec{x}; \|\vec{x} - \vec{a}\| < r\}$. (or " \leq "). Not trivial to prove!
(A ball is also an "interval" -esque set.)

• The set of solutions of $\vec{A} \vec{x} = \vec{b}$.
Proof: If \vec{u}, \vec{v} are sol'n, then $\vec{A} (\lambda \vec{u} + (1-\lambda) \vec{v}) = \lambda \vec{b} + (1-\lambda) \vec{b}$
is therefore a solution.

• Exercises: The set of $\vec{x}; \vec{A} \vec{x}$ is coordinate-wise $\leq \vec{b}$

• Note: The solutions of $\vec{A} \vec{x} = \vec{b}$: just like
lin. comb's, lin. indep./dep., the concept works way
beyond n -vectors!

Intersecting convex sets

Claim: The intersection $S = S_1 \cap S_2$ of two convex sets S_1 and S_2 , is a convex set.

Proof: Suppose $\vec{u} \in S$, $\vec{v} \in S$ and $\lambda \in (0, 1)$.

Then $\lambda \vec{u} + (1-\lambda) \vec{v}$ is: $\begin{cases} \in S_1, \text{ as both points are } \in S_1 \text{ and } S_1 \text{ is convex} \\ \in S_2, \text{ same argument.} \end{cases}$

and thus $\in S_1 \cap S_2$.

The book calls this proof "One of the world's simplest". If you don't think it is simple, it is probably a language hurdle: draw a Venn diagram!


could be more than a sequence!

Exercise: Let $\{S_j\}_{j \in J}$ be an arbitrary collection of convex sets. Show that $\bigcap_{j \in J} S_j$ is convex.

Terminology: the convex hull of a set T

is the intersection of all convex supersets.

Fact: it is the smallest convex superset of T ,
by prev. exercise possibly = T

Ex.: $T =$ this curve in \mathbb{R}^2 : 

Convex hull:  drop-shaped.

Note: the union of convexes need not be

convex. E.g.: 

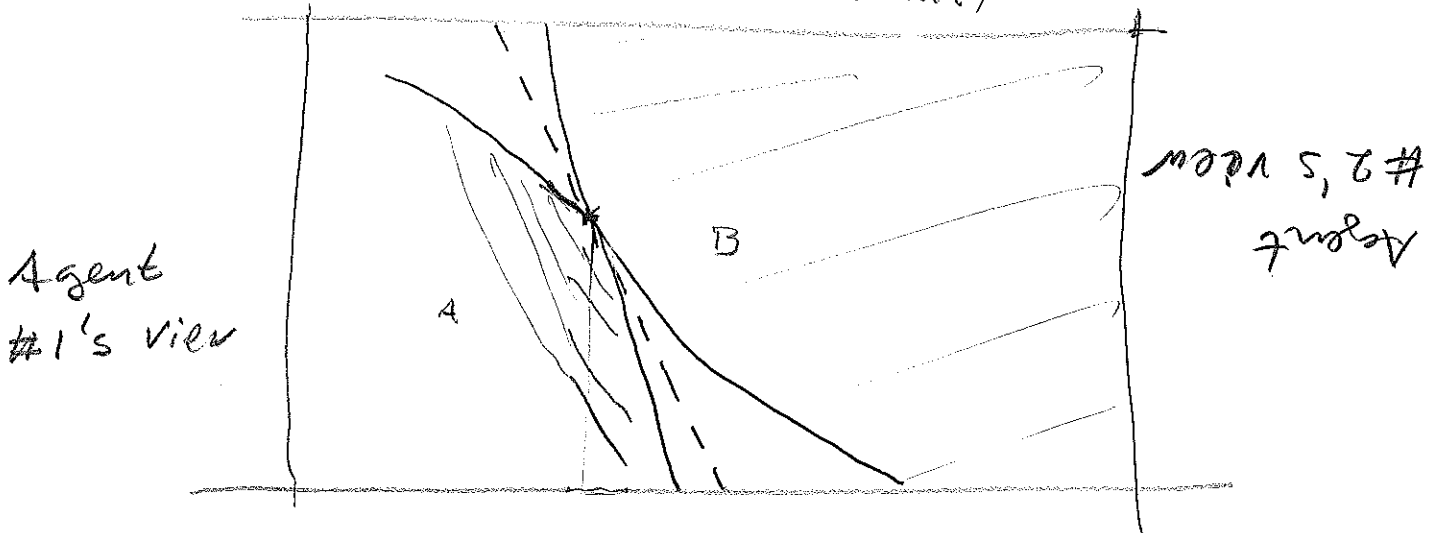
Why convex sets?

Separating hyperplanes & the 2nd theorem of welfare economics

Convex preferences \leftrightarrow the set of \vec{x} s.t

$$\vec{x} \in \bigcup_{\text{weak Pref}} \vec{x}^* \text{ convex, each } \vec{x}^*$$

Edgeworth box, 2 goods, 2 agents (in bounded amount)



Fact: Given two disjoint convex sets A, B , A open. Then there is a \vec{p} such that $\vec{p}^T \vec{u} < \vec{p}^T \vec{v}$, all $\vec{u} \in A, \vec{v} \in B$. (Dashed line: $\vec{p}^T \vec{x} = m$, budget...)

but works way more generally!
 $\underline{e} \in \mathbb{R}^n$

(The pricing functional \vec{p} is not unique. If boundaries are "smooth at tangency", \vec{p} is unique up to scaling.)

Convex preferences \leftrightarrow

quasiconcave utility functions

First: the more restrictive notion
of convex / concave functions.

Convex / concave functions

Note: "functions" here output numbers, like Math 2.

We shall give two defns for convex functions.

First, a geometric.

Terminology: Epigraph: set of points on or above the graph.

$\text{epi}(f)$: the set $\{(\vec{x}, z); z \geq f(\vec{x})\}$.

Ex: $f(x) = \max\{0, x\}$



"fill up with water"

Note: one more dimension than \vec{x} .

Definition (I): convex function.

Let f be defined on some convex $S \subseteq \mathbb{R}^n$
or: some vector space!

f is convex if $\text{epi}(f)$ is a convex set.

and strictly convex if $\text{epi}(f)$ is a strictly convex set.

Note: The requirement that S be convex, follows automatically if we omit it.

Ex: $|x|$ is convex. Even $\|\vec{x}\|$ (not completely obvious!)

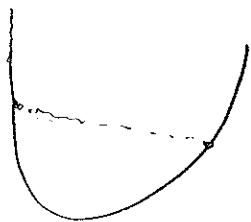
Definition (II), convex function (defined on convex S):

f is convex if, whenever $\vec{u} \in S, \vec{v} \in S, \lambda \in (0, 1)$

$$f(\lambda \vec{u} + (1-\lambda) \vec{v}) \leq \lambda f(\vec{u}) + (1-\lambda) f(\vec{v}), \quad (*)$$

f is strictly convex if $(*)$ holds with $<$ except when $\vec{u} = \vec{v}$.

Def. II:

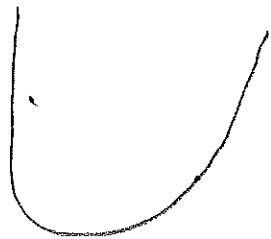


Pick two points on the graph. The connecting line segment is never below the graph.

("always above" for strict).

Equivalent def's!

Def I



Pick two points on or above the graph. The connecting ...

Concave functions:

Def: f is concave if $-f$ is convex.

f is strictly concave if $-f$ is strictly convex.

We can then formulate def's analogous to the above ^{two}.

Non-strict versions:

I: the convexity of the hypograph, i.e. the set of points on or below the graph

II: Whenever $\vec{u} \in S$, $\vec{v} \in S$ and $\lambda \in (0,1)$:

$$f(\lambda \vec{u} + (1-\lambda)\vec{v}) \geq \lambda f(\vec{u}) + (1-\lambda)f(\vec{v})$$

Note: For each pair \vec{u}, \vec{v} of points - assume $\vec{u} \neq \vec{v}$ -

consider $h(\lambda) = f(\lambda \vec{u} + (1-\lambda)\vec{v}) - \lambda f(\vec{u}) - (1-\lambda)f(\vec{v})$

"Convenient:" single variable $\lambda \in (0,1)$.

Similar for convex & for "strict"

II': f concave \Leftrightarrow for each pair $\vec{u} \neq \vec{v}$ in S we have $h \geq 0$ on the interval $(0,1)$ (and... h concave in λ !)

We could consider convex comb's of several vectors:

f concave if
 $f(\text{convex comb. of } \vec{v}^{(i)}) \geq$
convex comb. of the $f(\vec{v}^{(i)})$.

and even Jensen's inequality: f concave

" \Leftrightarrow " $f(E \vec{Y}) \geq E f(\vec{Y})$ "all" random vectors \vec{Y}

but then we would have to add a reservation like "as long as convergent or $= +\infty$ or $= -\infty$ " ...

Jensen's inequality ties concavity to

risk aversion: $f(E Y) \geq E f(Y)$ says

"prefer (weakly) the expectation to the r.v."

But, let's stick to def's I and II.

Proving concavity/convexity from def's
could be demanding.

Ex: $f(x) = |x|$.

$$h(\lambda) = |\lambda u + (1-\lambda)v| - \lambda|u| - (1-\lambda)|v|.$$

No restriction to assume $u > v$

We have:

$$h'(\lambda) = (u-v) \operatorname{sign}(\lambda u + (1-\lambda)v) - (|u| - |v|)$$

smallest at $\lambda=0^+$, since $v < u$
if changes sign:
from - to +.

So h' nondecreasing, with a possible upwards jump.

Since $h(0^+) = h(1^-) = 0$, we must "have the decreasing part of h first", so $h \leq 0$.

This idea works also for "Math 2 convex" functions.

Suppose $g'' \geq 0$. Let $v < u$.

$$\text{Then } h(\lambda) = g(\lambda u + (1-\lambda)v) - \lambda g(u) - (1-\lambda)g(v)$$

$$h'(\lambda) = (u-v)g'(\lambda u + (1-\lambda)v) - g(u) + g(v)$$

$$h''(\lambda) = (u-v)^2 g''(\lambda u + (1-\lambda)v) \geq 0.$$

So h' nondecreasing and h starts and ends at 0;

h must have the decreasing part first,

so $h \leq 0$ and g convex. (def II).

Properties ... ? Characterization?

- * Tempting to start generally, then impose conditions ... "if $f \in C^1$ " or "if $f \in C^2$."
(Shouldn't I then rather have started with quasiconvexes / quasiconcaves?)
 - * Alternative: C^2 first, then C^1 , then ...
 - * Will do: First a few general properties that do not use derivatives (but could, if applicable)
Then: characterization for C^2 functions
Then: characterization for C^1 functions
Then: characterization if not even C^1
- If you prefer a different order - reshuffle the notes!

Three general facts:

Concave?

1) If f and g are convex (on same domain) }
then $\max\{f(\vec{x}), g(\vec{x})\}$ is convex }
} $\min\{f, g\}$ concave.

(Indeed: works for more than two functions!)

2) If f, g convex and $\alpha > 0, \beta > 0$, }
then $\alpha f + \beta g$ convex }
} and strictly convex if f or g is }
} [α: different domains?] } ditto!

3) Consider $F(\vec{x}) = h(f(\vec{x}))$
where f convex, takes values in $T \subseteq \mathbb{R}$
 h convex and nondecreasing on T .
Then F is convex
Concave version: if f, h concave and h nondecreasing then F concave.

Before proving: how do 1) - 3) relate to Math 2?

1) the Math 2 case

well, in Math 2 we would need $\max\{f, g\} \in C^2$
as well, so then it would be a nonneg 2^{nd} deriv.
(concave: $\min\{...\}$... nonpos ...)

2) the Math 2 case

At least for functions of a single variable,
 $\alpha f'' + \beta g'' \dots \geq 0$!

3) the Math 2 case:

Easy for functions of a single variable

$$F' = h'(f(x)) f'(x)$$
$$F''(x) = h''(f(x)) \underbrace{(f'(x))^2}_{\geq 0} + \underbrace{h'(f(x))}_{\text{assumed } \geq 0} f''(x)$$

But generally?

Proof 1): intersect epigraphs!

Ex 1): $|x| = \max\{x, -x\}$ is convex!

Proof 2): Def'n II, apply ineq. for f and then g .

Strict case: one ineq strict!

$$\text{Proof 3) (convex): } f(\lambda \vec{w} + (1-\lambda) \vec{v}) \leq \lambda \overbrace{f(\vec{w})}^{=s} + (1-\lambda) \overbrace{f(\vec{v})}^{=t}$$
$$h(\quad) \leq h(\quad)$$

since h nondecr. RHS

$$\leq \lambda h(s) + (1-\lambda)h(t)$$

since h convex

Insert \dagger and we are done!

3) is "dangerous":

You cannot "just turn everything upside down" to switch between concaves and convexes.

Ex: $e^{\|\vec{x}\|}$ is convex.

$\|\vec{x}\|$ is convex, and e^t is convex & increasing

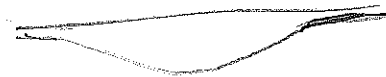
Non-ex:

$-e^{-\|\vec{x}\|}$ is not concave.

$-\|\vec{x}\|$ is concave, $-e^t$ is concave - but not nondecreasing!

($-e^{-\|\vec{x}\|} = -1$ and as $\|\vec{x}\| \rightarrow +\infty$,

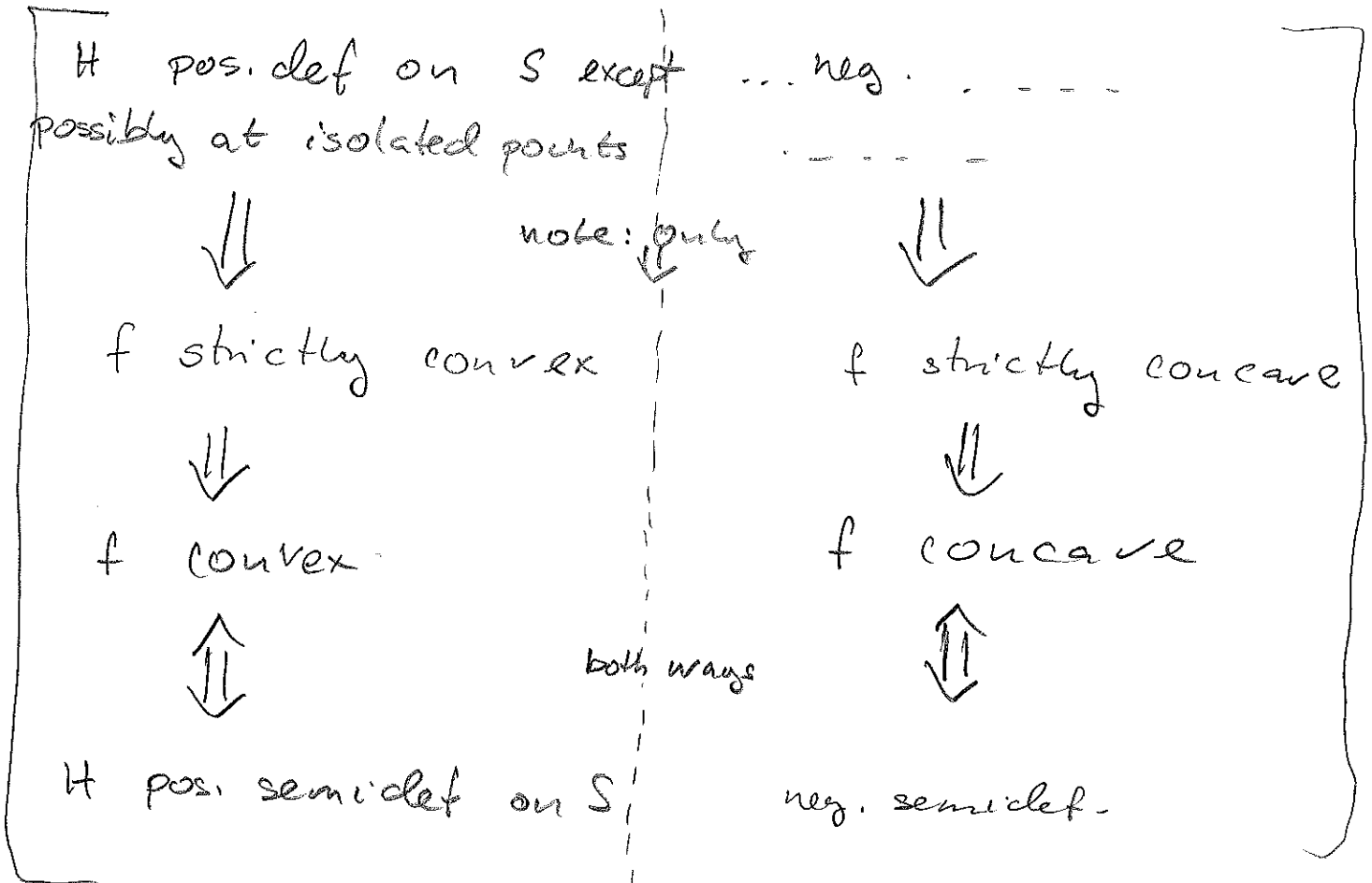
$-e^{-\|\vec{x}\|} \rightarrow 0$, so it cannot possibly be concave.



C^2 functions, defined on convex $S \subseteq \mathbb{R}^n$.

Let $\vec{H} = \vec{H}(x)$ be the Hessian matrix

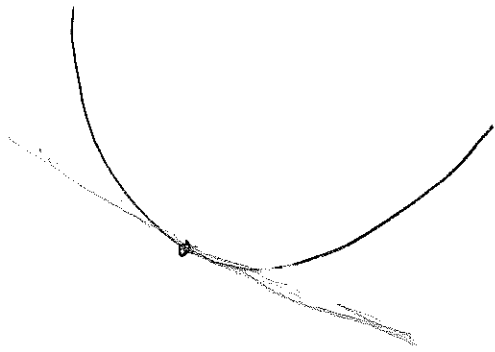
We have the implications



Note: the "except possibly": x^4 is strictly convex, yet Hessian hits 0.

(impossible for quadratic functions, Hessian matrix is constant)

- the first " \Downarrow " is not a \Uparrow .
det \vec{H} could hit zero "more often than that"
yet not destroying strict [concavity/convexity]

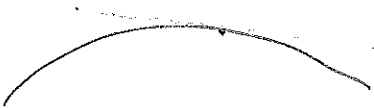


Convex:

Tangent never below the graph

- and touches only at that single point, for strictly convex functions.

1st order approx: underestimates



Concave:

Tangent never below graph.

- ... strict...

1st order approx: overestimates
around x^* $f(x^*)$.

Concave/convex functions are "not that far from being piecewise differentiable".

For C^1 functions - drop the C^2 assumption

A function $f \in C^1(S)$, S convex, $S \subseteq \mathbb{R}^n$,
open

is convex if and only if f is concave

$$f(\vec{x}) - f(\vec{x}^*) \geq \nabla f(\vec{x}^*) (\vec{x} - \vec{x}^*) \leq f(\vec{x}) - f(\vec{x}^*)$$

all $\vec{x} \neq \vec{x}^*$

(and strictly so iff the ineq is strict all $\vec{x} \neq \vec{x}^*$).

Note: adding linear terms will not change concavity/convexity, so we can alternatively write:

f convex iff the following holds:

For each $\vec{x}^* \in S$, the function g :

$$g(\vec{x}) = f(\vec{x}) - f(\vec{x}^*) - \nabla f(\vec{x}^*) (\vec{x} - \vec{x}^*)$$

↳
 "both or none convex"

has global min for $x = x^*$

• strictly if the min is always strict

For concave / strictly concave: "max"

Drop the C^1 assumption.

Fact: If the domain is open, a convex (concave) function is continuous.

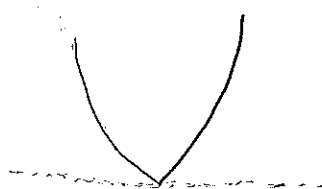
The only discontinuities can be on the boundary, (but they can be pretty bad):

$$\text{Let } S = \{(x, y) ; x^2 + y^2 \leq 1\}.$$

$$f = \begin{cases} \text{nonnegative strictly concave} \\ \text{on interior} \\ \text{anything} \leq 0, \text{ each boundary point} \end{cases}$$

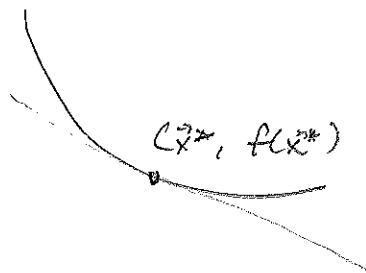
Yes, since S is a strictly convex set, we can draw $f(x, y)$ i'd random ≤ 0 at boundary points!

So: Assume continuity. Concave/convex functions will be continuous on $\mathbb{E}(\text{ON})4440$



But: can we say something sensible about this situation?

Recall from the C^1 case:



Terminology: Given f on open convex set S

Fix $\vec{x}^* \in S$. Consider affine functions z :

$$z - f(\vec{x}^*) = \vec{p} \cdot (\vec{x} - \vec{x}^*) \quad \text{in } \underbrace{S \times \mathbb{R}}$$

$z(\vec{x})$, affine

Same space as the graph of f ,
One dimension (on mod).

Def:

• If $z \leq f(\vec{x})$, all $\vec{x} \in S$, we

call \vec{p} a subgradient of f at \vec{x}^*

• The set of all such possible \vec{p} s.t. $z \leq f(\vec{x})$

is called the subdifferential of f at \vec{x}^*

Analogously: If $z \geq f(\vec{x}) \forall \vec{x} \in S$: supergradient

and the possible \vec{p} 's: superdifferential

Note that this defn is "pointwise", each \vec{x}^*

Facts: • If $\vec{\nabla} f(\vec{x}^*)$ exists, then it is the only possible sub-/supergradient at \vec{x}^*

• f convex \Leftrightarrow has a subgradient at each $\vec{x}^* \in S$

• concave \Leftrightarrow has a super.....

(Here we still assume f defined on open convex S .)

What did this mean ... ?

→ try to form subgradients for an arbitrary continuous f .

If f is not convex, you will fail somewhere ...

because z was required to be $\leq f$ everywhere, (each \vec{x}^*)



(This formulation of subgradients / subdiff. / super- ...)

is a "tailored to the narration" ... where the point is to grasp the geometric behaviour.)

So we have generalized the derivative.
Now generalizing stationary points ... ?

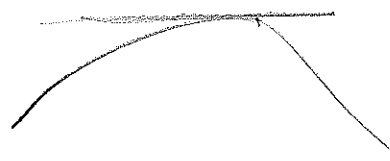
Note: a local min for a convex, is global
a local max ... concave

Fact: Let f be defined on an open convex S .

If $\vec{0}$ is a subgradient at x^* ,
then x^* is global min

If $\vec{0}$ is super- ...
global max

"Proof":



Put a horizontal hyperplane atop the graph.
If that is possible - without cutting through the graph elsewhere - then...

Also note:

The set of global min of a convex f
max concave

is convex.

(A "flat plateau atop a concave" must form a convex subset of S .
- this is a special case of quasi-concavity)

So... st. pts...?

C^1 functions: set $\nabla f = \vec{0}$

Convex/concave ... set ... what?

Sorry, Easy becomes inconvenient by hand (but not hopeless to implement numerically)

Simple example: $|x| + |y-3|$

$\nabla f = (\text{sign } x, \text{sign } (y-3))$ whenever well-defined

At $x=0$, $\frac{\partial f}{\partial x}$ crosses 0. At $y=3$, $\frac{\partial f}{\partial y}$ crosses 0.

So $\vec{0}$ does the subgradient job at $(0,3)$.

One more characterization:

Suppose f continuous on convex S .

If for every pair \vec{u}, \vec{v} in S , $\vec{u} \neq \vec{v}$
we have

$$f\left(\frac{1}{2}(\vec{u} + \vec{v})\right) \geq \frac{1}{2}(f(\vec{u}) + f(\vec{v}))$$

(resp \geq resp \leq resp $<$)

then f is convex resp strictly convex
resp concave resp strictly concave

So if we know we have continuity, then
the unweighted average $\lambda = \frac{1}{2}$ suffices!

Q: Is there anything "lost" by assuming
" S open and convex" rather than
" S convex, f continuous"?

A: Well... kinda...  vertical tangents,

Open S : tangents may \rightarrow vertical
slopes may $\rightarrow +\infty$ or $-\infty$

boundary: could be vertical ($\pm \infty$).

Do not bother.

Convex/concave functions of a single variable are integrals of their "derivatives".

Consider the following property.

⊗ Fix open interval $S = (a, b)$, There is a g such that for all α, β in S :

$$\int_{\alpha}^{\beta} g(x) dx = f(\beta) - f(\alpha).$$

Facts:

f convex on $(a, b) \Leftrightarrow$	\otimes holds with some nondecreasing g .
strictly convex	strictly increasing
concave	nonincreasing
strictly concave	strictly decreasing.

g could have infinitely many discontinuities, — for example, it could jump at every rational number!

... and recall: f convex \Leftrightarrow for any pair \vec{u}, \vec{v} the function $h(\lambda) = f(\lambda \vec{u} + (1-\lambda)\vec{v}) - \lambda f(\vec{u}) - (1-\lambda)f(\vec{v})$ is convex on $\lambda \in [0, 1]$