Seminar 17/4 Econ 4140.

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4-05. First we solve the "inner" integral:

$$\int_0^1 xe^y dy = x\int_0^1 e^y dy = x(e-1)$$

The we solve the "outer" integral:

$$\int_0^1 x(e-1)dx = (e-1)\int_0^1 xdx = (e-1)\frac{1}{2}$$

4-08. The integrand is $xy^2\cos(x^2y)$. Looks like something that could be the result of a chain rule differentiation. Let us try differentiating with respect to x.

$$\frac{d}{dx}\left(\sin\left(x^2y\right)\right) = \cos\left(x^2y\right)2xy$$

This looks promising. Let's look at the double integral.

$$\int_0^1\int_0^{\frac{\pi}{2}}\left(xy^2\cos\left(x^2y\right)\right)dydx = \int_0^{\frac{\pi}{2}}\int_0^1\left(xy^2\cos\left(x^2y\right)\right)dxdy$$

$$= \int_0^{\frac{\pi}{2}}\frac{1}{2}y\int_0^1\left(2xy\cos\left(x^2y\right)\right)dxdy$$

$$= \int_0^{\frac{\pi}{2}}\frac{1}{2}y\int_0^1\left(\frac{d}{dx}\left(\sin\left(x^2y\right)\right)\right)dxdy$$

$$= \int_0^{\frac{\pi}{2}}\frac{1}{2}y\left[\sin\left(x^2y\right)\right]_0^1 dy$$

$$= \int_0^{\frac{\pi}{2}}\frac{1}{2}y\sin\left(y\right)dy$$

To solve the last integral, use the formula $\int uv' = uv - \int u'v$. Let $u = ½y$ and let $v' = \sin(y)$. Then $v = -\cos(y)$, so

$$= \int_0^{\frac{\pi}{2}}\frac{1}{2}y\sin\left(y\right)dy = \left[-\frac{1}{2}y\cos\left(y\right)\right]_{y=0}^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{1}{2}\cos\left(y\right)dy = \frac{1}{2}$$

7-04 (a)

We have the equations:

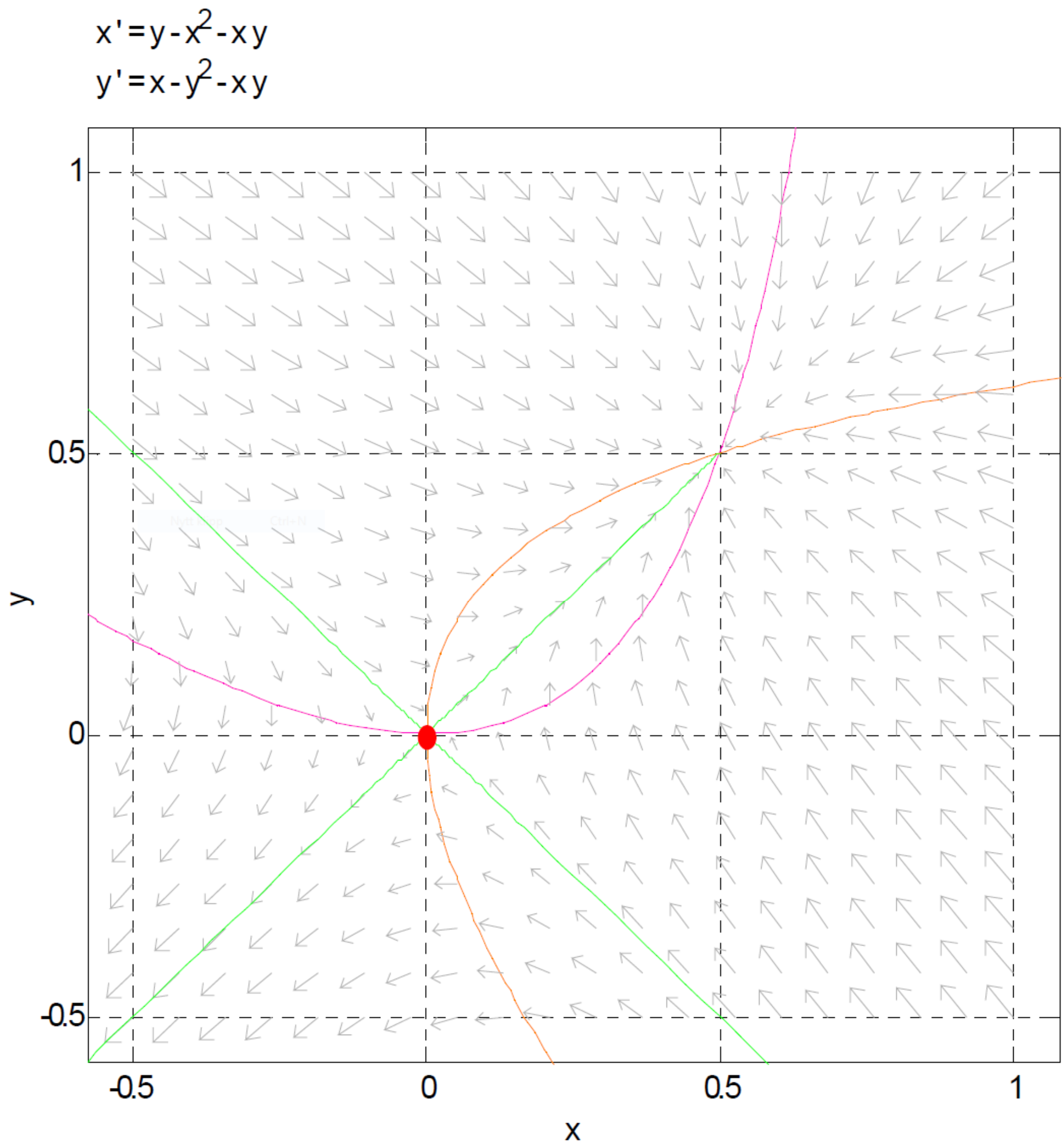
$$\dot{x} = y - x^2 - xy$$
$$\dot{y} = x - y^2 - xy$$

In order to find the equilibrium points we can either use our heads or we can start calculating. First we note that $x = y = 0$ is a solution. Then we see that x and y occur symmetrically in the two equations. It therefore makes sense to look for solutions where $x = y$ and $x = -y$. The first of these leads to $x = y = ½$. We calculate the matrix

$$\mathbf{A}\left(x, y\right) = \begin{bmatrix} -2x - y & 1 - x \\ 1 - y & -2y - x \end{bmatrix}$$

We can then calculate that the eigenvalues of $\mathbf{A}(0, 0)$ are $\lambda_1 = 1$ and $\lambda_2 = -1$. Since they have opposite sign, we have confirmed that [0, 0] is a saddle point. With $\mathbf{A}(½, ½)$ we do not need to calculate eigenvalues as $\mathrm{tr}(\mathbf{A}(½, ½)) = -3$ and $\det((\mathbf{A}(½, ½)) = 2$, so (½, ½) is locally asymptotically stable.

(b) A phase diagram looks like this:



Clearly (0, 0) is a saddle point and (½, ½) is locally stable.

(b) If $z = x + y$ then $\dot{z} = \dot{x} + \dot{y}$ so we can write the differentiable equation for z as follows

$$\begin{aligned}\dot{z} &= x + y - \left(x^2 + 2xy + y^2\right) \\ &= x + y - \left(x + y\right)^2 \\ &= z - z^2\end{aligned}$$

$$\begin{aligned}&\frac{\dot{z}}{z\left(1-z\right)} = 1 \\ &\left(\frac{1}{z} + \frac{1}{1-z}\right)\dot{z} = 1 \\ &\int\left(\frac{1}{z} + \frac{1}{1-z}\right)\dot{z}dt = \int 1dt \\ &\ln z - \ln\left(1-z\right) = t + K_0 \\ &\exp\left(\ln\frac{z}{1-z}\right) = K_1 e^t \\ &z = \frac{K_1 e^t}{1 + K_1 e^t}\end{aligned}$$

(c)

We use the hint. If $w = x - y$, then $\dot{w} = \dot{x} - \dot{y}$ which implies:

$$\dot{w} = -\left(x-y\right)\left(1 + x + y\right) \qquad (1)$$

As luck would have it we have already a solution for $x + y$. Inserting this expression into the differential equation for w gives:

$$-\frac{\dot{w}}{w} = \left(1 + \frac{K_1 e^t}{1 + K_1 e^t}\right)$$

The solution is straight forward to find:

$$
\begin{aligned}
&\int -\frac{\dot{w}}{w}dt = \int\left(1+\frac{K_1e^t}{1+K_1e^t}\right)dt\\
&\ln\frac{1}{w} = t + C_0 + \int\left(\frac{y'}{1+y}\right)dt\\
&\exp\left(\ln\frac{1}{w}\right) = \exp\left(t + C_0 + \ln\left(1+K_1e^t\right)\right)\\
&\frac{1}{w} = C_1e^t\left(1+K_1e^t\right)\\
&w = \frac{C_1e^{-t}}{\left(1+K_1e^t\right)}
\end{aligned}
$$

Now we have expressions for $x + y$ and $x - y$. We can use these to get expressions for x and y. Indeed, we have that:

$$
x+y=z=\frac{K_1e^t}{1+K_1e^t} \quad \text{and } x-y=w=\frac{C_1e^{-t}}{\left(1+K_1e^t\right)}
$$

Solving these expressions for x and y yields

$$
x\left(t\right)=\frac{e^tK_1+e^{-t}C_1}{2e^tK_1+2},\ y\left(t\right)=\frac{e^tK_1-e^{-t}C_1}{2e^tK_1+2}
$$

We are looking for particular solutions where $x(t) = y(t)$ for some particular t. The easiest way to bring about that is to set $C_1 = 0$. To find the particular solutions going through the given points, just fix the constant K_1. E.g:

$$
\frac{e^0K_1}{2e^0K_1+2}=1 \Rightarrow K_1=-2
$$

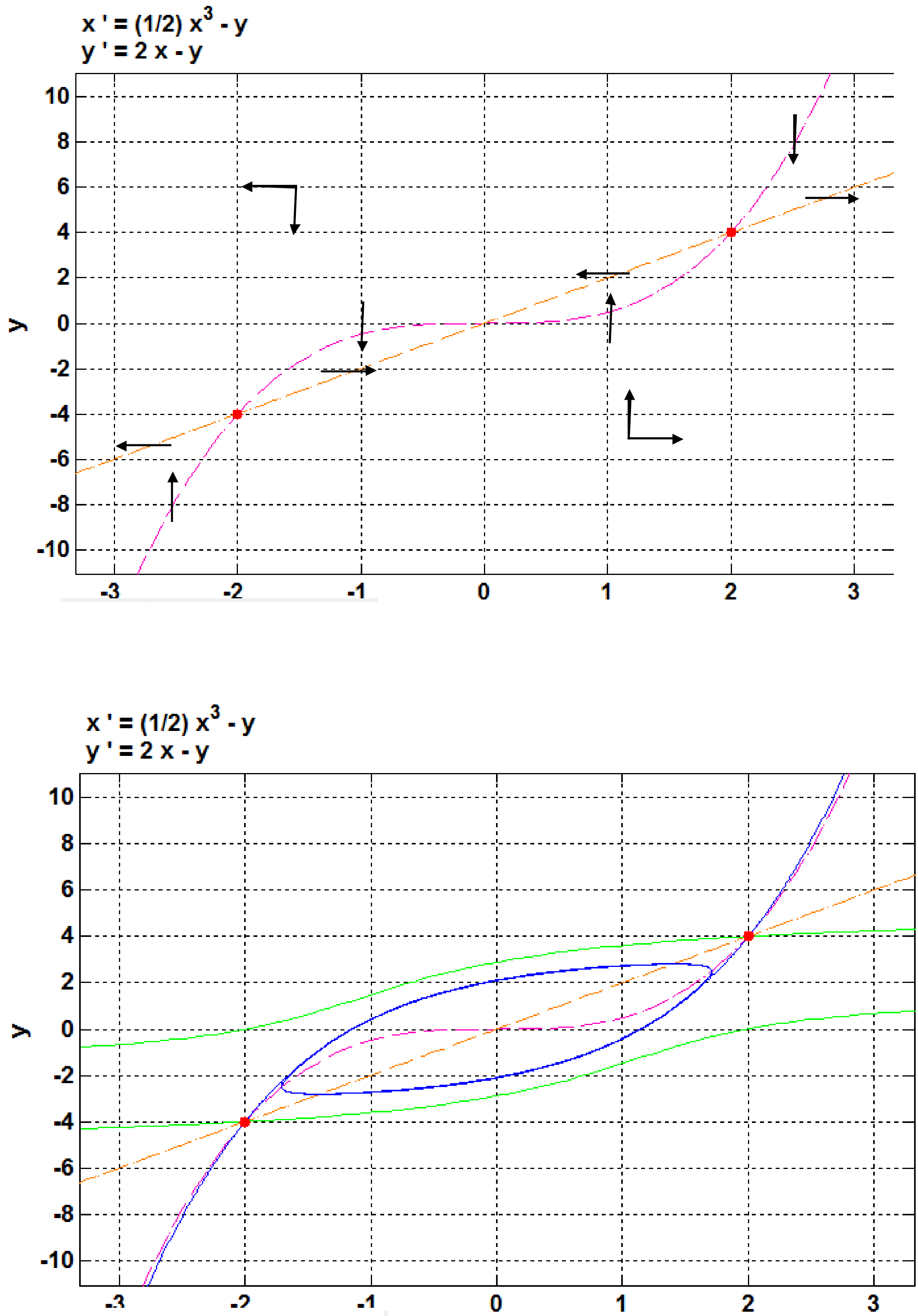
7-05

(a) Finding solutions to $\dot{x} = ½x^3 - y = 0$ and $\dot{y} = 2x - y = 0$ is easy. (-2, 4), (0, 0) and (2, 4) all fit. We find that the Jacobian is given by:

$$\mathbf{A}\left(x\right) = \begin{bmatrix} \frac{3}{2}x^2 & -1 \\ 2 & -1 \end{bmatrix}$$

It is straight forward to check that $\mathbf{A}(-2) = \begin{pmatrix} 6 & -1 \\ 2 & -1 \end{pmatrix}$ has eigenvalues of different signs so $(-2, -4)$ is a saddle point. Same goes for $\mathbf{A}(2) =$ as $\mathbf{A}(2) = \mathbf{A}(-2)$. Checking the eigenvalues of $\mathbf{A}(0)$ requires complex numbers. But since $\mathrm{tr}(\boldsymbol{A}(0)) = -1$ and $\det(\mathbf{A}(0)) = 2$, $(x, y) = (0, 0)$ is locally asymptotically stable. (In fact it is a spiral sink.)

(b) The phase diagram is drawn in two different versions below.



Figur 1. Here the blue lines indicate stable manifolds and green lines are unstable.

(c) We calculate the eigenvalues for $\mathbf{A}(2)$. They are:

$$\lambda_1 = \frac{1}{2}\left(5 - \sqrt{41}\right) \text{ and } \lambda_2 = \frac{1}{2}\left(5 + \sqrt{41}\right)$$

We only use λ_1. It follows that we can write $6x - y = \lambda_1 x$, which implies that $y = y = \frac{1}{2}\left(7 + \sqrt{41}\right)$.

8-01

Consider the variational problem

$$\max \int_0^1 (2xe^{-t} - 2x\dot{x} - \dot{x}^2)\, dt, \qquad x(0) = 0,\ x(1) = 1$$

(a) Write down the Euler equation for the problem.
(b) Find the solution of the problem, assuming it has one.

(a) We calculate:

$$\frac{\partial}{\partial x}\left(2xe^{-t} - 2x\dot{x} - \dot{x}^2\right) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(2xe^{-t} - 2x\dot{x} - \dot{x}^2\right)\right) = 0$$

$$2e^{-t} - 2\dot{x} - \frac{d}{dt}\left(-2x - 2\dot{x}\right) = 0$$

$$2e^{-t} - 2\dot{x} + 2\dot{x} + 2\ddot{x} = 0$$

$$2e^{-t} + 2\ddot{x} = 0$$

$$\ddot{x} = -e^{-t}$$

(b)

Solving the differential equation $\ddot{x} = -e^{-t}$ yields $x = -e^{-t} + Kt + C$, where K and C are arbitrary constants. Fixing these from initial and endpoint conditions implies:

$$x\left(0\right) = -1 + C = 0,\ x\left(1\right) = -e^{-t} + K + C = 1$$

It follows that $C = 1$ and $K = e^{-1}$ so $x = -e^{-t} + e^{-1}t + 1$.

8-06

(a) Vi skal maksimere $\int_{t_0}^{t_1}\left(px^2+\frac{q}{b^2}\left(\dot{x}-x\right)^2\right)dt$. The Euler equation is given by:

$$\frac{\partial}{\partial x}\left(px^2 + \frac{q}{b^2}\left(\dot{x} - x\right)^2\right) + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(px^2 + \frac{q}{b^2}\left(\dot{x} - x\right)^2\right)\right) =$$

$$2px(t) - \frac{2aq\left(x'(t) - ax(t)\right)}{b^2} - \frac{2q\left(x''(t) - ax'(t)\right)}{b^2} =$$

$$\ddot{x} - \left(a^2 + \frac{b^2 p}{q}\right)x(t) = 0$$

This is straightforward. The characteristic equation is given by:

$$r^2 - \left(a^2 + \frac{b^2 p}{q}\right) = 0$$

The solution is clearly $r_{1,2} = \pm\left(a^2 + \frac{b^2 p}{q}\right)$ and $x = Ae^{r_1 t} + Be^{r_2 t}$.

Of course, if $r_i = 0$, the solution is even simpler.

(b) Inserting the numbers gives us the differential equation

$\ddot{x} - x = 0$, implying that $r_{1,2} = \pm 1$. Thus we have that

$$x = Ae^{t} + Be^{-t}$$

Letting $x(0) = 0$ and $x(1) = 1$ gives the equations

$$A + B = 0, \;\; Ae + \frac{B}{e} = 1$$

These are straighforward to solve and gives the solution:

$$x(t) = \frac{e^{1+t} - e^{1-t}}{e^2 - 1}$$

9-03

We form the Hamiltonian $H = x - u^2 + (x + u)$. Clearly H is concave in x and strictly concave in u. The maximum principle yields the following conditions:

$$u = \frac{\mu}{2}$$
$$\dot{\mu} = -1 - \mu,\ \ \mu(2) = 0$$
$$\dot{x} = x + \frac{\mu}{2}$$

Solving the differential equation for yields:

$$\dot{\mu} + \mu = -1$$
$$\dot{\mu} e^t = \int -e^t dt$$
$$\mu = -1 + Ke^{-t}$$

Using the transversality condition yields $-1 + Ke^{-2} = 0 \rightarrow K = e^2$. Thus $u = ½(e^{2-t} - 1)$. In order to find the optimal x we solve

$$\dot{x} = x + ½(e^{2-t} - 1)$$

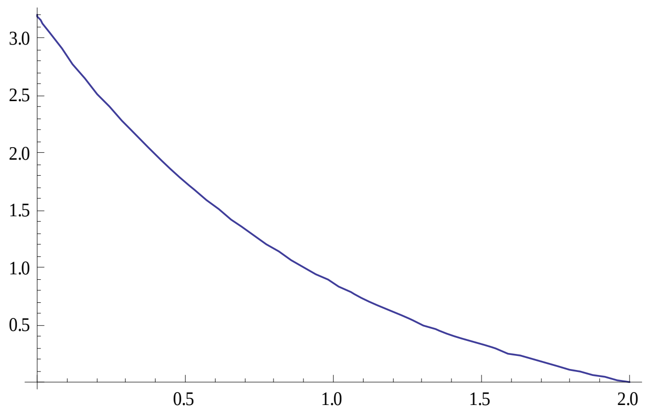
Solving this equations gives the equation:

$$
\begin{aligned}
\dot{x} - x &= \tfrac{1}{2}\left(e^{2-t} - 1\right) \\
xe^{-t} &= \frac{1}{2}\int\left(e^{2-2t} - e^{-t}\right)dt \\
xe^{-t} &= -\frac{1}{4}e^{2-2t} + \frac{1}{2}e^{-t} + K \\
x &= -\frac{1}{4}e^{2-t} + \frac{1}{2} + Ke^{t}
\end{aligned}
$$

Then $x(0) = 0$ implies that $K = \frac{1}{4}\left(-2 + e^{2}\right)$.

b)

Optimality conditions are the same except that we must pay attention to the constraints on u. However, the differential equation for μ does not change so $\mu(t)$ is still given by $\mu(t) = e^{2-t} - 1$. If we plot $\mu(t)/2$ we get something like this:



Clearly, the constraint $u \leq 1$ is binding for low t. We can calculate that $p(t)/2 \geq 1$ implies that $t \leq t^* = 2 - \ln 3$.

We now have to recalculate $x(t)$ over two intervals $[0, t^*]$ and $[t^*, 2]$. The solution over the first interval is found by solving

$$\dot{x} = x + u = x + 1, \ x\left(0\right) = 0$$

The solution is given by $x\left(t\right) = e^{t} - 1$. We calculate that $x(2 - \ln 3) = \mathrm{Exp}(2)/3 - 1$. Then we need to find $x(t)$ over $(t^*, 2]$. This we can do by solving the following differential equation:

$$\dot{x} = x + ½(e^{2-t} - 1), \quad x\left(2 - \ln 3\right) = \frac{e^2}{3} - 1$$

The solution to this equation is

$$x(t) = \frac{1}{2} - \frac{e^{2-t}}{4} - \frac{9e^{t-2}}{4} + e^t$$