ECON 4140, Seminar April 6th.

A

Find the antiderivative of $t\cos t$.

We use integration by parts. Let $t = v$ and let $\cos t = u'$, which implies that $u = \sin t$. Then

$$\int t\cos t dt = t\sin t - \int \sin t dt$$
$$= t\sin t + \cos t$$

B

Solve the differential equation $x'' - 4x' + 5x = 0$

We form the characteristic equation $r^2 - 4r - 5 = 0$. This has the solutions $r = -1$ and $r = 5$. Thus the solution is:

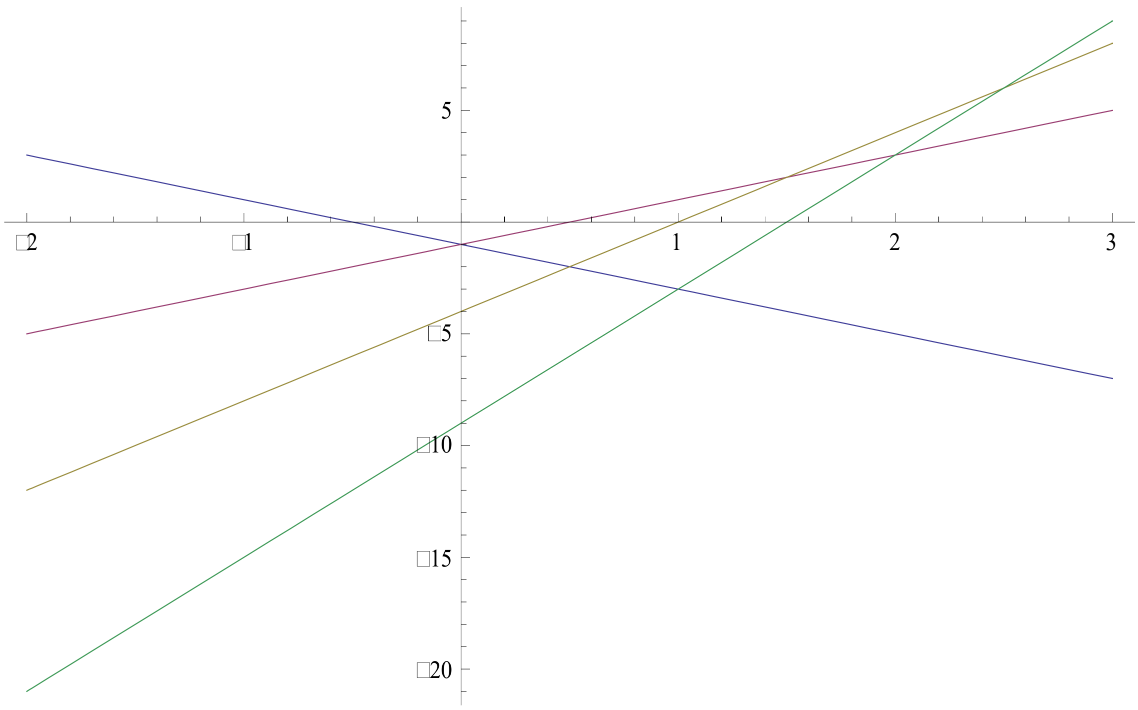
$$x\left(t\right) = C_1 e^{-rt} + C_2 e^{5t}$$

Sometimes it is possible to verify quasiconcavity/quasiconvexity by checking the level curves directly etc.

a)

Let $f(x, y) = x + (x^2 - y)^{½}$. A mechanical way of finding level curves is to solve $f(x, y) = C$ with respect to e.g. y. This gives y

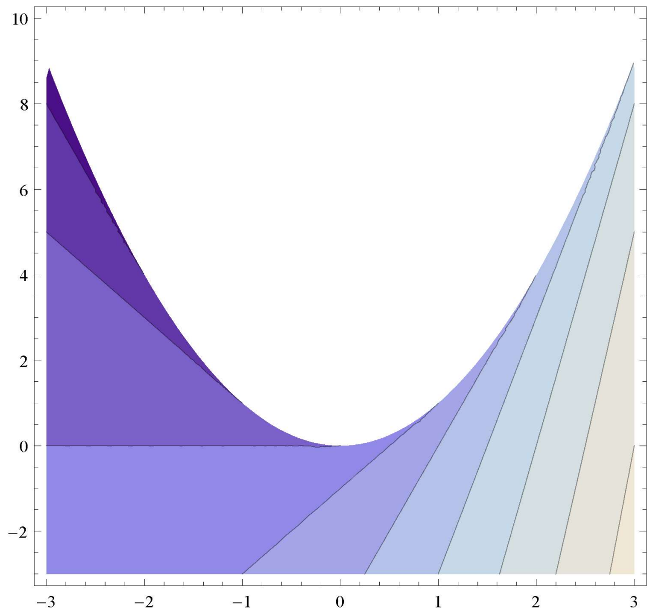
$= 2cx - c^2$. Plotting this expression for $c = -1, 1, 2$ and 3 gives the following plot:



Here we immediately see that something is wrong. These lines can not be level curves. (Why?)

b) If we look at $f(x, y)$ again we see that it is only defined when $y \leq x^2$. But that means that $f(x, y)$ is not defined over a convex set, which means that the definition of quasi-concavity/convexity is not satisfied so it makes no sense to ask the question in the first place. Below is a drawing of what the level curves look like when that fact is taken into account.

Obviously we can ensure that quasi-concavity/convexity holds by removing the set of values $f(x, y)$ is defined over. If we insist that $(x, y) \in A = \{(x, y): x \in R, y \leq 0\}$ we have quasi-concavity/convexity over A.



6-12

a) We have the following, rather messy equation:

$$\dot{p}(t) = \beta \int_{-\infty}^{t} \left[D(p(\tau)) - S(p(\tau))\right] e^{-\alpha(t-\tau)}\, d\tau$$

$D(p) = a - bp$ and $S(p) = -c + dp$. In order to derive a 2. orders differential equation from an expression where you only have the first order derivative it makes sense to take the derivative. With the aid of Leibnitz' formula:

$$\ddot{p} = \beta\left(\left(a+c\right)-\left(b+d\right)p\left(t\right)\right)e^{-\alpha\left(t-t\right)}$$
$$+\beta\int_{-\infty}^{t}\frac{d}{dt}\left(\left(\left(a+c\right)-\left(b+d\right)p\left(\tau\right)\right)e^{-\alpha\left(t-\tau\right)}\right)d\tau$$
$$\ddot{p} = \beta\left(\left(a+c\right)-\left(b+d\right)p\left(t\right)\right)-\alpha\beta\int_{-\infty}^{t}\left(\left(a+c\right)-\left(b+d\right)p\left(\tau\right)\right)e^{-\alpha\left(t-\tau\right)}d\tau$$
$$\ddot{p} = \beta\left(\left(a+c\right)-\left(b+d\right)p\left(t\right)\right)-\alpha\dot{p}$$
$$\ddot{p}+\alpha\dot{p}+\beta\left(b+d\right)p\left(t\right)=\beta\left(a+c\right)$$

b) Equilibrium implies $\ddot{p} = \dot{p} = 0$ which implies that $p^* = \frac{a+c;b+d}{}$.

c)

Let us check the possibilities. If $\frac{1}{4}\alpha^2 - \beta(d + b) > 0$, then the characteristic equation has two real roots given by

$$r_1 = -\frac{1}{2}\alpha + \sqrt{\tfrac{1}{4}\alpha^2 - \beta\left(b+d\right)}, \quad r_2 = -\frac{1}{2}\alpha - \sqrt{\tfrac{1}{4}\alpha^2 - \beta\left(b+d\right)}$$

r_2 is clearly negative. r_1 is also negative as long as $\beta(b + d) > 0$. Thus $A\exp(r_1 t) + B\exp(r_2 t)$ both goes to zero as $t \to \infty$. The two other cases are easier to check. They will go to zero as long as $-\frac{1}{2}\alpha < 0$. Of course we can just sum this up in the observation that the equation $\ddot{x} + a\dot{x} + b = f\left(t\right)$ is stable as long as $a > 0$ and $b > 0$. Oscillations will only occur if $\frac{1}{4}\alpha^2 - \beta(d + b) < 0$.

6-15

a) First we write the problem as

$$\ddot{x} + 4\dot{x} + 13x = \frac{1}{2}e^{2t}$$

Using formulas from the book we have that $\frac{1}{4}a^2 - b = 4 - 13 < 0$ so the homogenous solution is on the form $x = e^{\alpha t}(A\times\cos(\beta t) + B\times\sin(\beta t))$ where $\alpha = -\frac{1}{2}a = -2$ and $\beta = \sqrt{b - \frac{1}{4}a^2} = \sqrt{13 - \frac{1}{4}4^2} = 3$. Then we find the particular solution. The book tells us to try a solution of the form $u = Ae^{2t}$. This yields $u' = 2Ae^{2t}$ and $u'' = 4Ae^{2t}$. Inserting into the equation yields

$$4Ae^{2t} + 8Ae^{2t} + 13Ae^{2t} = \frac{1}{2}e^{2t}$$

$$A = \frac{1}{50}$$

b) We now try to find a particular solution when

$$\ddot{x} + 4\dot{x} + 13x = \frac{1}{2}\sin\left(3t\right)$$

We try to fit a solution of the form $u = A\times\sin(3t) + B\times\cos(3t)$. This gives us $u' = 3A\times\cos(3t) - 3B\times\sin(3t)$ and $u'' = -9B\cos(3t) - 9A\sin(3t)$. Inserting this into the differential equation yields:

$$4\left(3A + B\right)\cos(3t) + 4\left(A - 3B\right)\sin(3t) = \frac{1}{2}\sin\left(3t\right)$$

This gives us the equations $12A + 4B = 0$ and $4A - 12B = \frac{1}{2}$ with the solution $A = 1/80$ and $B = -3/80$.

7-01.

We have the equations:

$$\dot{x} = x + y + t, \quad \dot{y} = -x + 2y$$

Taking derivatives of both yields:

$$\ddot{x} = \dot{x} + \dot{y} + 1, \quad \ddot{y} = -\dot{x} + 2\dot{y}$$

We can now think of these equations as a system with 4 equations and 7 unknowns. We may solve for y, $\dot{y}$ and $\ddot{x}$ as functions of $\dot{x}$, x og t. In particular we are interested in finding $\ddot{x}$ as a function of $\dot{x}$, x og t. We can write the answer as:

$$\ddot{x} - 3\dot{x} + 3x = 1 - 2t$$

The characteristic equation is $r^2 - 3r + 3 = 0$. This equation only has complex solutions. The homogenous equation therefore has the solution $x = e^{\alpha t}(A\times\cos(\beta t) + B\times\sin(\beta t)$ hvor $\alpha = -½\times 3 = -3/2$ and $\beta = \sqrt{3 - ¼3^2} = \sqrt{3}\,/\,2$. In order to find the particular solution we try a guess that $u = Ct + D$. Then $u' = C$ and $u'' = 0$. Inserting into the equation yields

$$0 - 3C + 3\left(Ct + D\right) = 1 - 2t$$
$$3Ct = -2t, \ 3C + 3D = 1$$

implying $C = -2/3$ og $D = -1/3$. Therefore $x(t)$ is given by:

$$x = \exp\left(\frac{3}{2}t\right)\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\left(\frac{\sqrt{3}}{2}t\right)\right) - \frac{2}{3}t - \frac{1}{3}$$

We also need to find $y(t)$. We have that: $\dot{x} - x - t = y$. Taking the derivative of $x(t)$ yields:

$$x'\left(t\right) = \frac{1}{2}e^{3t/2}\left(\left(3A + \sqrt{3}B\right)\cos\left(\frac{\sqrt{3}t}{2}\right) + \left(3B - \sqrt{3}A\right)\sin\left(\frac{\sqrt{3}t}{2}\right)\right) - \frac{2}{3}$$

Inserting $x(t)$ and $x'\left(t\right)$ yields, after an ungodly amount of algebra:

$$y = \frac{1}{2}e^{3t/2}\left(\left(A + \sqrt{3}B\right)\cos\left(\frac{\sqrt{3}t}{2}\right) + \left(B - \sqrt{3}A\right)\sin\left(\frac{\sqrt{3}t}{2}\right)\right) - \frac{1}{3}(t + 1)$$

7-02

a) We have the system:

$$\begin{aligned}\dot{x} &= ax + 2y + \alpha \\ \dot{y} &= 2x + ay + \beta\end{aligned} \qquad (*)$$

This system may be solved in several ways, the easiest being to use eigenvalues. Here we will transform the system to a second order equation. We differentiate (*) and get:

$$\begin{aligned}\ddot{x} &= a\dot{x} + 2\dot{y} \\ \ddot{y} &= 2\dot{x} + a\dot{y}\end{aligned} \qquad (**)$$

(*) and (**) are 4 equations and 6 variables. We can therefore solve for $\ddot{x}$ as a function of $\dot{x}$ and x. We then get

$$\ddot{x} - 2a\dot{x} - \left(4 - a^2\right)x = 2\beta - a\alpha \qquad (***)$$

The characteristic equation is $r^2 - 2a - (4 - a^2) = 0$. This equation has real solutions for all values of a. They are given by $r_1 = a + 2$ and $a - 2$. Thus the homogenous equation has the solution $x = Ae^{(2+a)t} + Be^{(2-a)t}$. This solution to the homogenous equation is valid for all values of a. In order to find the particular solution we try a solution on the form $u = K$. It is easy to see that this implies that $u = \left(2\beta - a\alpha\right)/\left(a^2 - 4\right)$. Thus the full solution for x is given by:

$$x\left(t\right) = Ae^{\left(a+2\right)t} + Be^{\left(a-2\right)t} + \frac{\left(2\beta - a\alpha\right)}{a^2 - 4}$$

We now proceed to find $y(t)$. The solution for $x(t)$ implies that:

$$\dot{x} = x'\left(t\right) = A\left(a+2\right)e^{\left(a+2\right)t} + B\left(a-2\right)e^{\left(a-2\right)t}$$

Inserting into the equation for $\dot{x}$ above gives:

$$\dot{x} = ax + 2y + \alpha$$
$$\Downarrow$$
$$A(a+2)e^{(a+2)t} + B(a-2)e^{(a-2)t} =$$
$$a\left(Ae^{(a+2)t} + Be^{(a-2)t} + \frac{(2\beta - a\alpha)}{a^2 - 4}\right) + 2y(t) + \alpha$$
$$\Downarrow$$
$$y(t) = Ae^{(a+2)t} + Be^{(a-2)t} + \frac{2\alpha - a\beta}{a^2 - 4}$$

These solutions are clearly only valid when $a \neq \pm 2$. We therefore have to look for other particular solutions in this case. If $a = \pm 2$, we can write (***) as:

$$\ddot{x} - 2a\dot{x} = 2\beta - a\alpha$$

By solving this equation as a first order differential equation and then integrating we get the solution:

$$u = K_2 + \frac{K_1}{2a}e^{2at} + \frac{a\alpha - 2\beta}{2a}t$$

Here K_1 and K_2 are arbitrary constants. Now let $a = 2$. Then the general solution is given by:

$$x = Ae^{4t} + B + K_2 + \frac{K_1}{4}e^{4t} + \frac{\alpha - \beta}{2}t$$

By writing $A + \frac{1}{4}K_1 = \Gamma_1$ and $B + K_2 = \Gamma_2$, the general solution may be written:

$$x = \Gamma_2 + \Gamma_2 e^{4t} + \frac{\alpha - \beta}{2}t$$

Here Γ_1 and Γ_2 are arbitrary constants. Finding y is now plain algebra and the answer is given.

Now let $a = -2$. Then the general solution is

$$x = A + Be^{-4t} + K_2 - \frac{K}{4}e^{-4t} + \frac{\alpha + \beta}{2}t$$

Again we can rewrite by letting $B - \frac{1}{4}K_1 = \Gamma_1$ and $A + K_2 = \Gamma_2$ resulting in

$$x = \Gamma_2 + \Gamma_1 e^{-4t} + \frac{\alpha - \beta}{2}t$$

Again I leave the task of finding $y(t)$ to you.

b)

We already have the equilibrium points from above. If $x'(t) = x''(t) = 0$, then A and B must be chosen so that:

$$x = \frac{(2\beta - a\alpha)}{a^2 - 4},\quad y = \frac{2\alpha - a\beta}{a^2 - 4}$$

Alternatively we could just solve the system:

$$\dot{x} = ax + 2y + \alpha = 0$$
$$\dot{y} = 2x + ay + \beta = 0$$

Written in matrix form (*) may be written:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$
$$= \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

This system is only stable if $\mathrm{tr}(\mathbf{A}) < 0$ and $\det(\mathbf{A}) > 0$. This requires $a < -2$. If $-2 < a < 2$, the steady state is a saddle point as A has eigenvalues with opposite signs.

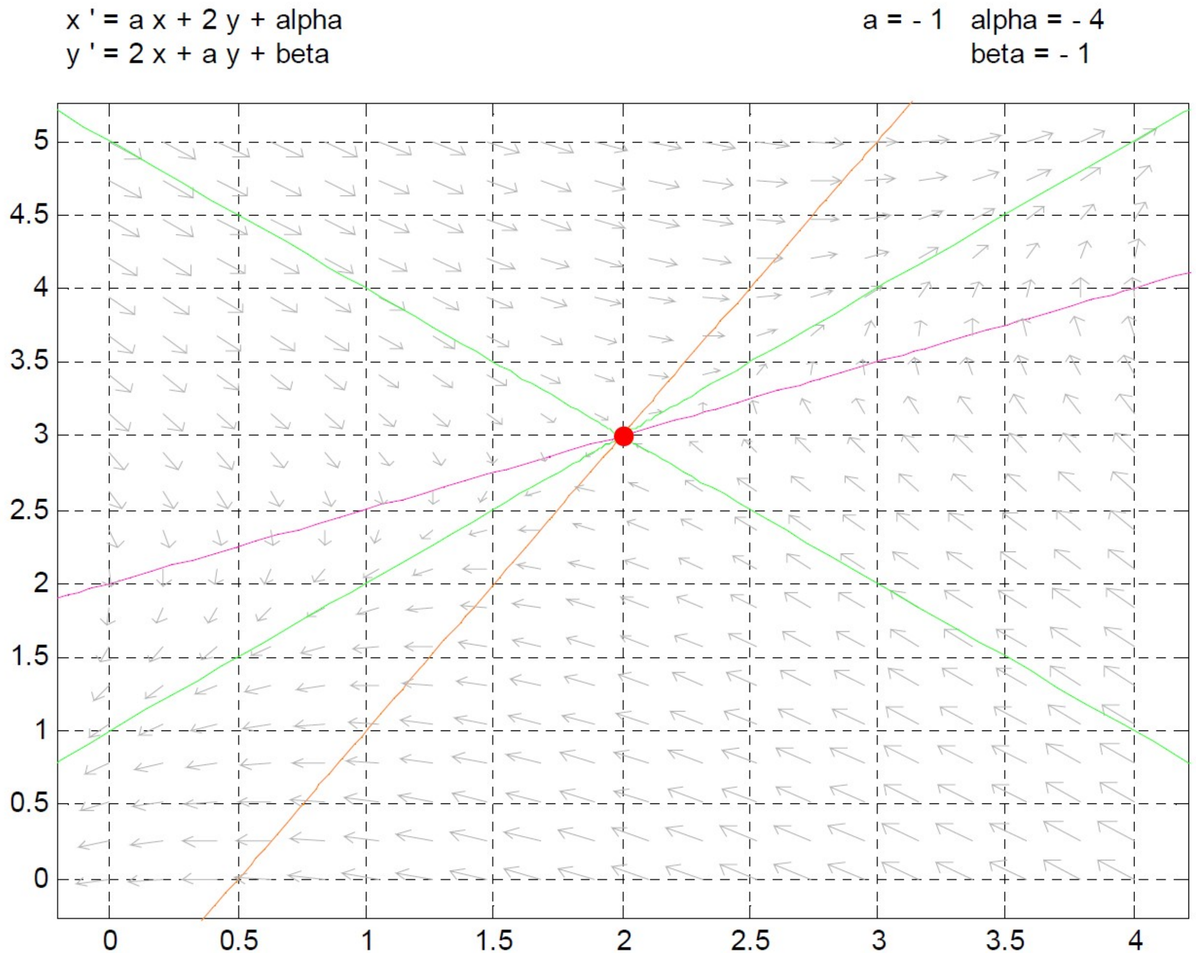
d)

Draw the lines $-x + 2y - 4 \geq 0 \rightarrow| \; y \geq ½x + 2$ and $2x - y - 1 \geq 0$

$\rightarrow y \leq 2x - 1$. As $a = -1$, we know that any equlibrium point

must be a saddle point. As I am the teacher I can use a

computer to draw the actual diagram.



Classify the origin as equilibrium point for the system etc.

We have the differential equations:

$$\dot{x} = 1 - e^{x-y},\ \ \dot{y} = -y$$

We further have that

$$\mathbf{J}\left(x,y\right) = \begin{bmatrix} \dfrac{d\dot{x}}{dx} & \dfrac{d\dot{x}}{dy} \\ \dfrac{d\dot{y}}{dx} & \dfrac{d\dot{y}}{dy} \end{bmatrix} = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 0 & -1 \end{bmatrix}$$

There is only one equilibrium point $(x,\ y) = (0,\ 0)$. We know that locally around the equilibrium point the solution can be approximated by:

(*)
$$\mathbf{J}\left(0,0\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The stability properties of the original system are locally the same as the stability properties of (*) around the equilibrium point. It is straight forward to calculate that $\mathrm{Tr}(\mathbf{J}(0,\ 0)) = –2 < 0$ and $\det(\mathbf{J}(0,\ 0)) = 1 > 0$, so by Liapunovs's Theorem the system is locally stable.

It is straight forward to see that $\mathrm{Tr}(\mathbf{J}(x,\ y)) < 0$ and $\det(\mathbf{J}(x,\ y)) > 0$ for all $(x,\ y)$, så by Olech's theorem the system is globally stable.