Seminar 9/2.

Positive definiteness requires that **x'Ax** > 0 and positive semidefiniteness requires **x'Bx** $\geq$ 0 for all **x**. Therefore

$$\mathbf{x}'\mathbf{A}\mathbf{x} + \mathbf{x}'\mathbf{B}\mathbf{x} > 0$$
$$\mathbf{x}'\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}\right) > 0$$
$$\mathbf{x}'\left(\mathbf{A} + \mathbf{B}\right)\mathbf{x} > 0$$

The sum of **A** and **B** is therefore positive definite.

How about the product of a positive definite matrix and a positive semidefinite matrix? Let:

$$\mathbf{A} = \begin{bmatrix} a_1 & A \\ A & a_2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_1 & B \\ B & b_2 \end{bmatrix}$$

In order for **A** to be positive definite, a_1 and $|\mathbf{A}|$ must be strictly positive. A can be negative as long as A^2 is less than $a_1 a_2$. If we now calculate **AB** we find that:

$$\mathbf{AB} = \begin{bmatrix} a_1 b_1 + AB & a_1 B + A b_2 \\ A b_1 + a_2 B & AB + a_2 b_2 \end{bmatrix}$$

Clearly **AB** is not symmetric and therefore not definite in any way. However the quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{ABx}$ has an equivalent form with a matrix that is symmetric.

Compendium problem 1-13.

a) We have

$$\left(\mathbf{X}+\tfrac{1}{2}\mathbf{A}^{-1}\mathbf{B}'\right)'\mathbf{A}\left(\mathbf{X}+\tfrac{1}{2}\mathbf{A}^{-1}\mathbf{B}'\right)-\tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}'$$

$$\left(\mathbf{X}'+\left(\tfrac{1}{2}\mathbf{A}^{-1}\mathbf{B}'\right)'\right)\left(\mathbf{A}\mathbf{X}+\tfrac{1}{2}\mathbf{B}'\right)-\tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}'$$

$$\left(\mathbf{X}'+\left(\tfrac{1}{2}\mathbf{B}\left(\mathbf{A}^{-1}\right)'\right)\right)\left(\mathbf{A}\mathbf{X}+\tfrac{1}{2}\mathbf{B}'\right)-\tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}'$$

$$\mathbf{X}'\mathbf{A}\mathbf{X}+\tfrac{1}{2}\mathbf{X}'\mathbf{B}'+\left(\tfrac{1}{2}\mathbf{B}\left(\mathbf{A}^{-1}\right)'\right)\mathbf{A}\mathbf{X}+\left(\tfrac{1}{2}\mathbf{B}\left(\mathbf{A}^{-1}\right)'\right)\tfrac{1}{2}\mathbf{B}'-\tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}'$$

$$\mathbf{X}'\mathbf{A}\mathbf{X}+\tfrac{1}{2}\mathbf{X}'\mathbf{B}'+\tfrac{1}{2}\mathbf{B}\mathbf{X}+\tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}'-\tfrac{1}{4}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}'$$

$$\mathbf{X}'\mathbf{A}\mathbf{X}+\mathbf{B}\mathbf{X}$$

b) We have

$$\begin{aligned}\frac{d}{d\mathbf{X}}\left(\mathbf{X}'\mathbf{A}\mathbf{X}+\mathbf{B}\mathbf{X}\right)&=\mathbf{X}'\mathbf{A}+\mathbf{A}\mathbf{X}+\mathbf{B}\\&=\mathbf{A}\mathbf{X}+\mathbf{A}\mathbf{X}+\mathbf{B}\\&=2\mathbf{A}\mathbf{X}+\mathbf{B}=\mathbf{0}\\&\Downarrow\\\mathbf{X}&=-\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\end{aligned}$$

That this is a minimum follows from **A** being positive definite.

Exam 2008 problem 1(c)

$$\mathbf{A}=\begin{bmatrix}0&2&0\\2&3&0\\0&0&0\end{bmatrix}$$

We have that:

$$D_1=|0|$$

This implies that A is not positive definite. We have that:

$$D_2=\det\begin{vmatrix}0&2\\2&3\end{vmatrix}=-4$$

This is one of the principal subdeterminants, so A is not positive semidefinite. As $D_1=0$ we can rule out negative definiteness. For $r=1$, the principal subdeterminants are all non-negative so $(-1)^1\Delta_r$ = [0 -3 0] so we can also rule out negative semidefiniteness.

Alternatively: Note that **A** is symmetric. Then we can use eigenvalues. It is straight forward to calculate that the eigenvalues for **A** are given by -1, 0 and 4. With eigenvalues of different signs we now that A is indefinite.

Now we add the constraint that

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 16 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 16 & 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 variables and two constraints. We must therefore calculate the following determinant.

$$D_3 = \begin{vmatrix} 0 & 0 & 16 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 16 & 1 & 0 & 2 & 0 \\ 0 & 2 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = -5$$

Fist we check if it is positive definite. We note that there are $m = 2$ constraints and calculate that $(-1)^m D_3$ = -5. Thus the constrained quadratic form is not positive definite. Then we calculate $(-1)^3 D_3 = 5$ and the constrained quadratic form is negative definite.

Differentiation

We have that $g:\mathbb{R}^m \to \mathbb{R}$. We have that the Hessian is $\mathbf{H}(\mathbf{y})$ where the element in the ith row and the jth column is $g_{i,j}(\mathbf{y})$. We are asked to find the Hessian of $f(\mathbf{x}) = g(\mathbf{Ax})$. Don't think to hard about this stuff. Just use the chain rule. First we get the gradient.

$$\mathbf{h}(\mathbf{x}) = \nabla f(\mathbf{Ax}) = \left[\sum_{i=1}^{m} g_i'(\mathbf{Ax})a_{i,1}, \sum_{i=1}^{m} g_i'(\mathbf{Ax})a_{i,2}, \cdots, \sum_{i=1}^{m} g_i'(\mathbf{Ax})a_{i,n}\right] = \underbrace{\left[\underbrace{\nabla g(\mathbf{Ax})}_{1\times m} \underbrace{\mathbf{A}}_{m\times n}\right]}_{1\times n}$$

$$= \underbrace{\mathbf{A}^{\mathbf{T}}}_{n\times m} \underbrace{\nabla g(\mathbf{Ax})^{\mathbf{T}}}_{m\times`1}$$

Then take the derivative of this expression.

$$\underbrace{\mathbf{h}'(x)}_{n\times n} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \mathbf{A}^{\mathbf{T}} \underbrace{\begin{bmatrix} g_{11}''(\mathbf{Ax}) & g_{12}''(\mathbf{Ax}) & \cdots & g_{1m}''(\mathbf{Ax}) \\ g_{21}''(\mathbf{Ax}) & g_{22}''(\mathbf{Ax}) & \cdots & g_{2m}''(\mathbf{Ax}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{m1}''(\mathbf{Ax}) & g_{m2}''(\mathbf{Ax}) & \cdots & g_{12}''(\mathbf{Ax}) \end{bmatrix}}_{m\times m} \mathbf{A}$$

$$= \mathbf{A}^{\mathbf{T}}\mathbf{H}(\mathbf{Ax})\mathbf{A}$$

Problem 2-07

Let $f(x,y) = (\ln x)^a(\ln y)^b$, defined for $x > 1$, $y > 1$. Assume that $a > 0$, $b > 0$ and $a + b < 1$. Compute the Hessian matrix $\mathbf{H}$ of f and show that f is strictly concave.

We have that

$$f''_{x,x}\left(x,y\right)=\frac{-(1-a)a\left(\ln x\right)^{a-2}\left(\ln y\right)^{b}}{x^2}-\frac{a\left(\ln x\right)^{a-1}\left(\ln\left(y\right)\right)^{b}}{x^2}$$
$$f''_{y,y}\left(x,y\right)=\frac{-(1-b)b\left(\ln x\right)^{a}\left(\ln y\right)^{b-2}}{y^2}-\frac{b\left(\ln x\right)^{a}\left(\ln y\right)^{b-1}}{y^2}$$
$$f''_{x,y}\left(x,y\right)=\frac{ab\left(\ln x\right)^{a-1}\left(\ln y\right)^{b-1}}{xy}$$

$f''_{x,x}$ is clearly negative. The Hessian is easy to form after having calculated these expressions. Noting that $f''_{x,x}\left(x,y\right)$ and $f''_{y,y}\left(x,y\right)$ has common factors simplify the calculations. The determinant of the Hessian is

$$\begin{aligned}\left|\mathbf{H}\right|&=\frac{ab\left(\ln x\right)^{2a-2}\left(\ln y\right)^{2b-2}\left(1-a+\ln x\right)\left(1-b+\ln y\right)}{x^2y^2}-\frac{a^2b^2\left(\ln x\right)^{2a-2}\left(\ln y\right)^{2b-2}}{x^2y^2}\\&=\frac{ab}{x^2y^2}\left(\ln x\right)^{2a-2}\left(\ln y\right)^{2b-2}\left(\left(1-a+\ln x\right)\left(1-b+\ln y\right)-ab\right)\\&=\frac{ab}{x^2y^2}\left(\ln x\right)^{2a-2}\left(\ln y\right)^{2b-2}\underbrace{\left(1-\left(a+b\right)+\left(1-a\right)\ln y+\left(1-b\right)\ln x+\ln\left(x+y\right)\right)}_{\text{positive by assumption}}>0\end{aligned}$$