Seminar 20/4. Econ 4140. Eric Nævdal

9-05

Problem 9-05

Consider the variational problem

$$\max \int_0^T (ax^2 + 2bx\dot{x} + c\dot{x}^2 + dt^2\dot{x})e^{-rt}\,dt, \qquad x(0) = x_0, \quad x(T) = x_T \qquad (*)$$

(a) For what values of the constants a, b, c, d, and r is $(ax^2 + 2bxy + cy^2 + dt^2y)e^{-rt}$ concave with respect to (x, y)?

(b) Find the Euler equation associated with $(*)$.

(c) Solve the problem

$$\max \int_0^1 (-9x^2 + 2x\dot{x} - \dot{x}^2 + 3t^2\dot{x})\,dt, \qquad x(0) = 0, \quad x(1) = 0 \qquad (**)$$

(You can use the result in (b).)

(d) Transform the problem $(**)$ in (c) into a control problem and find the optimal solution when the terminal condition is changed from $x(1) = 0$ to

(i) $x(1)$ free, (ii) $x(1) \geq 2$.

a) Define $F(x,y) = (ax^2 + 2bxy + cy^2 + dt^2y)e^{-rt}$. We have that $F''_{xx} = 2ae^{-rt}$, $F''_{yy} = 2c$ and $F''_{xx}F''_{yy} - F''^2_{xy} = 4ace^{-2rt} - 4b^2e^{-2rt}$. If F is to be either concave or convex, then $F''_{xx}F''_{yy} - F''_{xy} \geq 0$. This happens when $a \geq b^2/c$. If this holds concavity/convexity depends on the sign of a and c. sgn(a) = sgn(c) = 1, implies convexity. sgn(a) = sgn(c) = –1 implies concavity. sgn(a) =

$\text{sgn}(c) = 0$ implies both. $\text{sgn}(a) \neq \text{sgn}(b)$ implies neither. (Note: Both F''_{xx} and F''_{yy} must be checked as the exercise asks about concavity/convexity and not *strict* concavity/convexity.)

b) Calculating $F'_x\left(x,\dot{x}\right) - \frac{d}{dt}\left(F'_{\dot{x}}\left(x,\dot{x}\right)\right) = 0$ yields:

$$F'_x\left(x,\dot{x}\right) - \frac{d}{dt}\left(F'_{\dot{x}}\left(x,\dot{x}\right)\right) = 0$$
$$\Downarrow$$
$$\left(2ax(t) + 2bx'(t)\right)e^{-rt} - \frac{d}{dt}\left(\left(2bx(t) + 2cx'(t) + dt^2\right)e^{-rt}\right) = 0$$
$$\Downarrow$$
$$\left(2ax(t) + 2bx'(t)\right)e^{-rt} + e^{-rt}r\left(dt^2 + 2bx(t) + 2cx'(t)\right) - e^{-rt}\left(2dt + 2bx'(t) + 2cx''(t)\right) = 0$$
$$\Downarrow$$
$$dt(rt-2) + 2(a+br)x(t) + 2crx'(t) - 2cx''(t) = 0$$
$$\Downarrow$$
$$x''(t) - rx'(t) - \frac{(a+br)}{2c}x(t) = \frac{\left(drt-2\right)t}{2c}$$

c)This corresponds to $a = –9$, $b = 1$, $c = 1$, $d = 3$ and $r = 0$.

Inserting yields:

$$\ddot{x} - 9x = 3t$$

The homogenous equation has the characteristic equation $r^2 – 9 = 0$ with the solution $r = \pm 3$. Also, if we try a particular solution $u = Ct$, we can determine that $C = –1/3$. so the general solution is $x = Ae^{3t} + Be^{–3t} – t/3$. $x(0) = x(1) = 0$ implies that:

$$\begin{aligned} &A + B = 0 \\ &Ae^3 + Be^{-3} - \tfrac{1}{3} = 0 \end{aligned}$$

Solving this yields:

$$A = \frac{1}{3\left(e^3 - e^{-3}\right)}, B = \frac{-1}{3\left(e^3 - e^{-3}\right)}$$

d)

First we set $\dot{x} = u \in \mathbb{R}$. The we have the problem:

$$\max_u \int_0^1 \left(-9x^2 + 2xu - u^2 + 3t^2 u\right) dt \ \ s.t : \dot{x} = u$$

The Hamiltonian is given by:

$$H = \left(-9x^2 + 2xu - u^2 + 3t^2 u\right) + pu$$

Note that H is concave in $(x,\ u)$. This leads us to the following optimality conditions.

$$u = \dot{x} = \frac{1}{2}\left(2x + 3t^2 + p\right)$$
$$\dot{p} = 18x - 2u$$

We differentiate this system and get

$$\ddot{x} = \frac{1}{2}\left(6t + \dot{p} + 2\dot{x}\right), \ \ \ddot{p} = -6t - \dot{p} + 16\dot{x}$$

We don't really need the last equation. Solving these four (or actually three) equations for $\ddot{x}$ yields

$$\ddot{x} - 9x = 3t$$

This should not be a big surprise. The same as what we got when we solved the calculus of variations problem. Obviously this equation has the same general solution. $x = Ae^{3t} + Be^{-3t} - t/3$. However we now have the condition that $x(1)$ is free. Thus we need a condition to fix the constants A and B. This condition is that

$$\left(\frac{\partial F}{\partial \dot{x}}\right)_{t=1} = -2x'\left(t\right) + 3 + 2x\left(t\right) = 0$$

Evaluating this expression is a bugger. It becomes

$$-4e^{3}A + 8Be^{-3} + 3 = 0$$

Together with $x(0) = A + B = 0$ we can fix A and B. They are given by:

$$A = \frac{3e^3}{4\left(2+e^6\right)}, B = -\frac{3e^3}{4\left(2+e^6\right)}$$

We now have to check what happens if we impose the condition $x(1) \geq 2$. Here it pays to stop and think. If the previous solution gives a solution where $x(1) > 2$, then we have already solved the problem. It turns out that $x(1)$ is now 0.411117. Thus we must work some more. However, we can use the conditions:

$$x\left(0\right) = A + B = 0$$
$$x\left(1\right) = Ae^3 + Be^{-3} - \tfrac{1}{3} = 2$$

Solving these gives:

$$A = \frac{-7e^3}{3\left(1-e^6\right)}, \quad B = \frac{7e^3}{3\left(1-e^6\right)}$$

Exam 2008 Exercise 1d)

let $\mathbf{A} = \mathbf{C}_0 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(d) Having previously calculated that there are three real and distinct eigenvalues $\lambda_1 = 4$, $\lambda_2 = -1$ and $\lambda_3 = 0$. With corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

It follows straight from formulas in the book that:

$$x(t) = C_1 e^{4t}\mathbf{v}_1 + C_2 e^{-t}\mathbf{v}_2 + C_3\mathbf{v}_3$$

We only need to look at $x(t)$. Clearly $C_1 = C_3 = 0$ for the system to approach the origin. If this holds, then $x_1(t) = -2C_2e^{-t}$ and $x_2(t) = C_2e^{-t}$ so $x_1(t)/x_2(t) = -2$.

Exam 2011

Problem 4 Let b be a given continuous function which is strictly positive for all t, and define $B(t)$ as $B(t) = \int_0^t b(s)\,ds$. Consider the optimal control problem

$$\max \int_0^T b(t) \ln u(t)\,dt \qquad \text{where} \quad \dot{x} = rx - u, \quad x(0) = x_0, \quad x(T) \geq 0$$

where r, x_0 and T are positive constants, and u is allowed to take any positive value.

(a) Show that the pair (x^*, u^*) defined by

$$u^*(t) = x_0 e^{rt} \frac{b(t)}{B(T)}$$
$$x^*(t) = x_0 e^{rt} \left(1 - \frac{B(t)}{B(T)}\right)$$

satisfies all the necessary conditions from the maximum principle.

(b) Show that (x^*, u^*) solves the problem.

Let $H = b(t)\ln u + p(rx - u)$. Then the Maximum Principle yields:

$$\frac{b(t)}{u(t)} - p(t) = 0$$
$$\dot{p} = -rp$$
$$p(T) \geq 0 \; (= 0 \text{ if } x(T) > 0)$$

The differential equation for p gives that $p = Ke^{-rt}$. This means that the condition that u maximizes the Hamiltonian may be written

$$u(t) = \frac{b(t)}{Ke^{-rt}}$$

Inserting for into the differential equation for x implies that we must solve:

$$\begin{aligned}
\dot{x} - rx &= -\frac{b(t)}{Ke^{-rt}} \\
\dot{x}e^{-rt} - rxe^{-rt} &= -\frac{b(t)}{Ke^{-rt}}e^{-rt} \\
\dot{x}e^{-rt} - rxe^{-rt} &= -\frac{b(t)}{K}
\end{aligned}$$

We now calculate the indeterminate integral on both sides of the equation:

$$\begin{aligned}
\int \left(\dot{x}e^{-rt} - rxe^{-rt}\right)dt &= \int -\frac{b(t)}{K}dt \\
xe^{-rt} &= -\frac{B(t)}{K} + C \\
x &= -\frac{B(t)}{K}e^{rt} + Ce^{rt}
\end{aligned}$$

We have two constants K and C to be determined. We can fix C from the condition $x(0) = x_0$. This gives

$$x_0 = -\frac{B(0)}{K} + C \Rightarrow C = x_0$$

We then have

$$x = -\frac{B(t)}{K}e^{rt} + x_0 e^{rt}$$

K will be fixed by the transversality condition. If the endpoint condition is binding so that $x(T) = 0$, we fix K in the following manner.

$$x(T) = -\frac{B(T)}{K}e^{rT} + x(0)e^{rT} = 0 \Rightarrow K = \frac{B(T)}{x(0)}$$

Inserting K into the solution above, $x(t)$ is given by:

$$x(t) = x(0)e^{-rt} - \frac{B(t)}{B(T)/x(0)} e^{rt} = x(0)e^{-rt}\left(1 - \frac{B(t)}{B(T)}\right)$$

Problem 3 Let $0 < K < Q < 1$ be constants and let G be a given function. Consider the differential equation system

$$\begin{aligned} \dot{x}(t) &= p(t) + Q \\ \dot{p}(t) &= Kx(t) - G(t) \end{aligned} \tag{D}$$

(a) Deduce a second-order differential equation for x, and find the general solution of this equation when $G \equiv 0$. (*Hint:* For which γ will $x(t) = e^{\gamma t}$ be a particular solution?)

(b) Find the general solution of (D) for the case when $G(t) = Ke^t$.

(a)

Calculating $\frac{d}{dt}(\dot{x}) = \ddot{x} = \dot{p} = Kx$ leads us to solve the following equation:

$$\ddot{x} - Kx = 0$$

The characteristic equation is $r^2 - K = 0$, which has the solution $r_t = -\sqrt{K}$ and $r_2 = \sqrt{K}$. Thus the general solution is

$$x = C_1 e^{-\sqrt{K}t} + C_2 e^{\sqrt{K}t}$$

b) We now look for a particular solution. Using the hint in the book we try a solution of the form Le^t. Then

$$\ddot{x} - Kx = Ke^t$$
$$\downarrow$$
$$Le^t - KLe^t = Ke^t$$
$$\downarrow$$
$$L = \frac{K}{1-K}$$

Thus

$$x(t) = C_1 e^{-\sqrt{K}t} + C_2 e^{\sqrt{K}t} + \frac{K}{1-K}e^t$$

$p(t)$ can then be calculated from $p = x'(t) - Q$

Problem 4 Let $0 < K < Q < 1$ be constants, and consider the optimal control problem

$$\max_{u(t)\in\mathbf{R}} \int_0^{11} \left\{ -\frac{K}{2} \cdot \left[x(t) - e^t\right]^2 - \frac{1}{2}\left[u(t)\right]^2 \right\} dt, \qquad \dot{x} = u + Q, \quad x(0) = x_0, \quad x(11) \text{ free.}$$

(a) i) State the conditions from the maximum principle.
ii) Are these conditions also sufficient?

(b) Show that in optimum, x and the adjoint (costate) p must satisfy the differential equation system (D) in problem 3, with $G(t) = Ke^t$.

(c) Suppose that for some set of parameters the optimal solution ends at $x(11) = 11e^{11}$. Approximately how much would the optimal *value* change if the final time were reduced from 11 to 10.9?

Quick and dirty

(a) i) simple. ii) Yes. Concave in x and u is enough for sufficiency.

(b) Simple

(c) Calculate $-H(T) \times 1/10$.