Seminar 220118 - Answers

1.2.4

4 three dimensional vectors are always linearly dependent. If vectors are linearly dependent we can find numbers x, y and z not all equal to zero such that:

$$x\begin{bmatrix}1\\1\\1\end{bmatrix}+b\begin{bmatrix}2\\1\\0\end{bmatrix}+c\begin{bmatrix}3\\1\\4\end{bmatrix}=\begin{bmatrix}1\\2\\-2\end{bmatrix}$$
$$\begin{bmatrix}1&2&3\\1&1&1\\1&0&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\-2\end{bmatrix}$$

This we can do if

$$\det\left(\begin{bmatrix}1&2&3\\1&1&1\\1&0&4\end{bmatrix}\right)\neq 0$$

Straightforward to show that determinant is -5. [x, y z] = [2, 1, –1].

1.4.1

a) The matrix

$$\begin{bmatrix} 1 & 2 \\ 8 & 16 \end{bmatrix}$$

The determinant is clearly zero so the vectors are linearly dependent, so rank is 1.

b) The matrix can be put in reduced form:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & -6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{7}{6} \end{bmatrix}$$

The two rows are clearly independent so rank is 2.

c) Again the matrix can be put in reduced form:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 2 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Obviously only 2 independent rows so rank is 2.

1.5.1

a) We have that

$$r\left(A\right) = r\left(\begin{bmatrix} -2 & -3 & 1 \\ 4 & 6 & -2 \end{bmatrix}\right) = r\left(\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\right) = 1$$
$$r\left(A_b\right) = r\left(\begin{bmatrix} -2 & -3 & 1 & 3 \\ 4 & 6 & -2 & 1 \end{bmatrix}\right) = r\left(\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 2$$

There are 2 equations and 2 variables. We have that $r(A) < r(A_b)$. We therefore have no solutions as the system is inconsistent.

b) We have that:

$$r\left(A\right)=r\left(\begin{bmatrix}1 & 1 & -1 & 1\\ 2 & -1 & 1 & -3\end{bmatrix}\right)=$$
$$r\left(\begin{bmatrix}1 & 1 & -1 & 1\\ 0 & -3 & 3 & -5\end{bmatrix}\right)=r\left(\begin{bmatrix}1 & 0 & 0 & -\frac{2}{3}\\ 0 & 1 & -1 & \frac{5}{3}\end{bmatrix}\right)=2$$
$$r\left(A_b\right)=r\left(\begin{bmatrix}1 & 1 & -1 & 1 & 2\\ 2 & -1 & 1 & -3 & 1\end{bmatrix}\right)$$
$$=r\left(\begin{bmatrix}1 & 1 & -1 & 1 & 2\\ 0 & -3 & 3 & -5 & 3\end{bmatrix}\right)=r\left(\begin{bmatrix}1 & 0 & 0 & -\frac{2}{3} & 1\\ 0 & 1 & -1 & \frac{5}{3} & 1\end{bmatrix}\right)=2$$

$r(A) = r(A_b) = 2$. Implies at least one solution. But as $k <$ the number of variables, 4, there are at two variables that can be freely chosen. La $x_3 = s$ og $x_4 = t$. Da blir

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1 & -\frac{5}{2}\\ 0 & \frac{2}{3}\end{bmatrix}\begin{bmatrix}x_3\\ x_4\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix} \tag{1}$$

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$$\mathbf{D}(s) = \begin{pmatrix} 1 & 2s & 1 & 1 \\ -2 & 1 & -2 & 3s \\ 1 & 1-s & -1 & 5 \\ -1 & 2 & s & -3 \end{pmatrix}$$

The determinant for this matrix is $10 + 15s - 16s^2 - 9s^3$ which must be unequal to zero for $r(\mathbf{D}(s)) = 4$. If $s = 1$ then the determinant is $10 + 15 - 16 - 9 = 0$. If we remove the last column and row we get::

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

The determinant for this matrix is -10, so $\mathbf{D}(1)$ has rank $= 3$.

(b)

Systemet har 4 likninger og 4 variable, men siden $\mathbf{D}(1)$ har rang 3 og **Db** har rank 3 så er det bare tre uavhengige likninger og systemet har derfor 1 frihetsgrad.

Oblig 1.

f)

We have a matrix

$$\mathbf{M} = \begin{pmatrix} 0.8 & 0 & 0.1 \\ 0 & 1.2 & -0.1 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic equation is given by:

$$\det\left(M - \lambda I_3\right) = \left(0.8 - \lambda\right)\left(1.2 - \lambda\right)\left(1 - \lambda\right)$$

Clearly the eigenvalues are $\lambda_1 = 0.8$, $\lambda_2 = 1.2$ and $\lambda_3 = 1$. To find corresponding eigenvectors, let us start with $\lambda = \lambda_1 = 0.8$. In this case $\mathbf{v}_1$ is defined by.

$$(\mathbf{M} - \lambda_1\mathbf{I}_3)\mathbf{v}_1 = \begin{bmatrix} 0 & 0 & .1 \\ 0 & .4 & -.1 \\ 0 & 0 & .2 \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly as all the numbers in the first column equals 0, any $\mathbf{v}_1$ satisfying this must be given by $s\times[\ 1,\ 0,\ 0]$ and s an arbitrary real number. Consider now $\lambda_2 = 1.2$. Then we have that

$$(\mathbf{M} - \lambda_2\mathbf{I}_3)\mathbf{v}_2 = \begin{bmatrix} -0.4 & 0 & .1 \\ 0 & 0 & -.1 \\ 0 & 0 & -.2 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we see that any vector $s\times[\ 0,\ 1,\ 0]$ and s an arbitrary real number makes the equation hold. Finally consider $\lambda_3 = 1$. Here we have that:

$$(\mathbf{M} - \lambda_3\mathbf{I}_3)\mathbf{v}_3 = \begin{bmatrix} -0.2 & 0 & .1 \\ 0 & 0.2 & -.1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we can choose x_3 freely as s. Then $[0,\ 0.2,\ -.1]\cdot[s/2,\ s/2,\ s] = 0$ implies that $[s/2,\ s/2,\ s]$ fits as an eigenvector.

g) The equilibrium state is simply $[x, y, s] = [0, 0, 0]$. This is not a globally stable equilibrium. However, if $[x_0, y_0, z_0] = [s, 0, 0]$ then the system converges as the trajectory is then given by $\mathbf{M}\cdot[s, 0, 0]' = 0.8^t \times [s, 0, 0]'[s, 0, 0]'$.

Any vector $[x_0, y_0, z_0] = [s/2, s/2, s]$ is also stable as $[x_t, y_t, z_t] = \mathbf{M}^t [s/2, s/2, s] = \lambda_1^t [s/2, s/2, s] = [s/2, s/2, s]$ for all t.

h) **M** is now given by:

$$\mathbf{M} = \begin{pmatrix} 0.8 & 0 & 0.1 \\ -0.1 & 1.2 & -0.1 \\ -0.1 & 0.1 & 1 \end{pmatrix}$$

The characteristic polynomial is given by:

$$\det\left(\begin{pmatrix} 0.8-\lambda & 0 & 0.1 \\ -0.1 & 1.2-\lambda & -0.1 \\ -0.1 & 0.1 & 1-\lambda \end{pmatrix}\right)$$
$$= (0.8-\lambda)(1.2-\lambda)(1-\lambda) - 0.1^3 + 0.1^2(1.2-\lambda) + 0.1^2(0.8-\lambda)$$

$\lambda_1 = 0.9$, $\lambda_2 = 1$ and $\lambda_3 = 1.1$ fits by inspection. One can show that this gives rise to the eigenvectors.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let us just calculate $\mathbf{v}_1$. When $\lambda = \lambda_1 = 0.9$, then:

$$(\mathbf{M} - \lambda_1 \mathbf{I}_3)\mathbf{v}_1 = 0 \Rightarrow \begin{pmatrix} -.1 & .1 & .1 \\ -.1 & .3 & -.1 \\ -.1 & .1 & .1 \end{pmatrix}\mathbf{v}_1$$

The third row is clearly superfluous, so we have 2 equations and 3 variables. One can be chosen freely. Let us choose x_3 as s. Then the first equation gives us

$$-x + y = -s$$
$$-x + 3y = s$$

Solving these equations for x and y gives that $x = 2s$ and $y = s$ verifying that $\mathbf{v}_1 = [2, 1, 1]'$.

j)

Let $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = [8, 5, 5]'$. Then let $\mathbf{V} = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3]$. Solve the equation $\mathbf{V}\cdot[a, b, c]' = [8, 5, 5]'$ and we have the solution. $[a, b, c] = [3, -2, 0]$.

k)

I think there is a misprint and it should be $M^n \cdot s_0$ and not $P^n \cdot s_0$ that the exercise wants us to find. This is found by:

$$\mathbf{M}^n s_0 = \mathbf{M}^n \left(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3\right)$$
$$= a\lambda_1^n \mathbf{v}_1 + b\lambda_2^n \mathbf{v}_2 + c\lambda_3^n \mathbf{v}_3$$