Econ 4140- Seminar 18/4-2017.

Problem 9-12

Find the only possible solution to the control problem

$$\max \int_0^2 (u^2 - x)\,dt \quad \text{s.t.} \quad \dot{x} = u, \;\; x(0) = 0, \;\; x(2) \text{ is free}, \;\; 0 \le u \le 1$$

that is, find the only admissible pair $(x(t), u(t))$ that satisfies the conditions in the maximum principle.

We form the Hamiltonian $H = u^2 - x + \mu u$. Clearly H is strictly convex in u and concave in x. As H is convex in u, the u that maximises the Hamiltonian is a corner solution, i.e. either $u(t) = 0$ or $u(t) = 1$. We start by calculating the adjoint variable:

$$\dot{\mu} = 1 \rightarrow \mu = K + t$$
$$x(2) \text{ free } \rightarrow \mu(2) = 0 \rightarrow K = -2$$

Thus $\mu(t) = t - 2$. We must now choose the optimal u. For arbitrary x and t we have that $H_{u=1} = 1 - x + (t-2)\times 1$ and $H_{u=0} = -x$. Thus choosing $u = 1$ is optimal if $H_{u=1} > H_{u=0}$. This happens if $t > 1$. If $t < 1$, it is optimal to let $u = 0$. At $t = 1$, we are indifferent. Thus $x'(t) = 0$ and $x(t) = 0$ for $t \in [0, 1)$.

$x'(t) = 1$ and $x(t) = t - 1$ for $t \in (1, 2]$.

The solution satisfies Arrow's sufficiency condition, so we know we have the optimal solution.

Induction.

Induction 1) Define $f(n) = \sum_{i=1}^{n} i$. Clearly $f(1) = 1$ and $f(2) = 2\times(2+1)/3 = 3$. If the hypothesis holds, then $f(n+1) = f(n) + n + 1 = n(n+1)/2 + n + 1 = (n+1)(n+2)/2$ which can be verified by straightforward algebra.

Induction 2). First we note that $9^1 - 1 = 8$. Then we calculate

$$\begin{aligned} 9^{n+1} &= 9^n\left(8+1\right) \\ 9^{n+1} - 1 &= \underbrace{9^n 8}_{\text{divisible by 8}} + \underbrace{9^n - 1}_{\substack{\text{divisible by 8 under} \\ \text{the hypothesis}}} \end{aligned}$$

10-03.

Fin the general solution to $x_{t+1} + x_{t+1} - 6x_t = 5^t + t$.

The characteristic equation is $r^2 + r - 6 = 0$. Straight forward to find the solution $r = 2$ and $r = -3$. The homogenous equation thus has the solution:

$$x_t = A2^t - B3^t$$

To find the particular solution we try a solution on the form $u = C5^t + Dt + F$. This gives that we can write $x_{t+1} + x_{t+1} - 6x_t = 5^t + t$ as

$$C5^{t+2} + D\left(t+2\right) + F + C5^{t+1} + D\left(t+1\right) + F - 6\left(C5^t + Dt + F\right) = 5^t + t$$
$$25C5^t + Dt + 2D + F + 5C5^t + Dt + D + F - 6C5^t - 6Dt + 6F = 5^t + t$$
$$\left(25C + 5C - 6C\right)5^t + \left(D + D - 6D\right)t + \left(2D + F + D + F - 6F\right) = 5^t + t$$

This implies that

$$25C + 5C - 6C = 1 \rightarrow C = \frac{1}{24}$$

$$D + D - 6D = 1 \rightarrow D = -\frac{1}{4}$$

$$2D + F + D + F - 6F = 0 \rightarrow 3D - 4F = 0 \rightarrow F = -\frac{3}{16}$$

10-05

(a) Solve the difference equation

$$x_{t+2} - \frac{5}{2}x_{t+1} + x_t = 10 \cdot 3^t, \qquad t = 0, 1, 2, \ldots$$

Determine the solution that gives $x_0 = 0$, $x_1 = 2$.

(b) The following difference equation appears in dynamic consumer theory:

$$-\alpha x_{t+1} + (1 + \alpha^2)x_t - \alpha x_{t-1} = K\beta^t, \qquad t = 1, 2, \ldots$$

Here α, K and β are constants, α and β positive. Determine the general solution of the equation when $\alpha \neq 1$, $\beta \neq \alpha$ and $\beta \neq 1/\alpha$.

(a) The characteristic equation has the solution $r = 2$ and $r =$ ½. So the solution to the homogenous equation is $A(½)^t + B2^t$. We try to find a particular solution on the form $D3^t$. Then we have that:

$$D3^{t+2} - \tfrac{5}{2}D3^{t+1} + D3^t = 10 \times 3^t$$

Solving this last expression with respect to D gives $D = 4$ so the general solution is given by:

$$x_t = A\left(\tfrac{1}{2}\right)^t + B2^t + 4 \times 3^t$$

In order to determine the constants A and B to find the solution going through $x_0 = 0$ and $x_1 = 2$, we calculate:

$$x_0 = A\left(\tfrac{1}{2}\right)^0 + B2^0 + 4 \times 3^0 = 0$$
$$x_t = A\left(\tfrac{1}{2}\right)^1 + B2^1 + 4 \times 3^1 = 2$$

(b)

A minor detail first. The equation depends on t +1, t and t – 1. Ignoring this *may* be unimportant. It certainly is for homogenous equations *and* many non-homogenous equations as well. To be on the safe side I use a transformation $\tau = t - 1$. Then we can write:

$$-\alpha x_{\tau+2} + \left(1 + \alpha^2\right) x_{\tau+1} - \alpha x_\tau = K\beta^{\tau+1}$$

The characteristic equation is given by:

$$m^2 - \frac{1+\alpha^2}{\alpha} m + 1 = 0$$

This equation has the solution $m_1 = \alpha$ and $m_2 = \alpha^{-1}$. In the process it may be helpful to note that $1 - 2\alpha^2 + \alpha^4 = (\alpha^2 - 1)^2$. Thus the solution to the homogenous equation is then $A\alpha^\tau + B\alpha^{-\tau}$. To find a particular solution, try with a solution on the form $u_\tau = D\beta^{\tau+1}$. Then the equation may be written:

$$-\alpha D\beta^{\tau+2} + \left(1+\alpha^2\right) D\beta^{\tau+1} - \alpha D\beta^{\tau} = K\beta^{\tau+1}$$

$$\downarrow$$

$$D = \frac{-\beta K}{\alpha + \alpha\beta^2 - \left(1+\alpha^2\right)\beta}$$

11-04

Consider the dynamic programming problem

$$\max \sum_{t=0}^{T-1} 2\sqrt{u_t x_t} + \sqrt{x_T} \quad \text{subject to} \quad x_{t+1} = x_t - u_t x_t, \quad u_t \in [0,1]$$

(Interpretation: My wealth today is $x_0 \geq 0$. Every day, i.e. for every $t < T$, I spend $u_t x_t$ dollars on consumption. On day T I spend the remaining wealth, x_T.

(a) Find $J_{T-1}(x)$ and $J_{T-2}(x)$ and the corresponding controls $u^*_{T-1}(x)$ and $u^*_{T-2}(x)$.

(b) Show that $J^*_t(x)$ can be written in the form $J^*_t(x) = k_t\sqrt{x}$, and find a difference-equation for k_t.

(a) Clearly $J_T(x) = \sqrt{x}$. Then $J_{T-1}(x) = \max_{u\in[0,1]}\left(2\sqrt{xu} + \sqrt{x - ux}\right) =$

$2\sqrt{\frac{4}{5}x} + \sqrt{\frac{1}{5}x} = \sqrt{5x}$ as $u = 4/5$. This because

$$\frac{\partial}{\partial u}\left(2\sqrt{xu} + \sqrt{x - ux}\right) = \frac{x}{\sqrt{ux}} - \frac{x}{2\sqrt{x - ux}} = 0$$

$$\downarrow$$

$$u = \frac{4}{5}$$

Then $J_{T-2}(x) = \max_{u\in[0,1]}\left(2\sqrt{xu} + \sqrt{5}\sqrt{x - ux}\right)$. Solving

$$\frac{\partial}{\partial u}\left(2\sqrt{xu} + \sqrt{5}\sqrt{x - ux}\right) = \frac{x}{\sqrt{ux}} - \frac{\sqrt{5}x}{2\sqrt{x - ux}} = 0$$

$$\downarrow$$

$$u = \frac{4}{9}$$

Inserting gives $J_{T-2}(x) = 3\sqrt{x}$.

b) This one seems horrible, but is a quite straightforward use of recursive math. We already know that for $T - 1$ the premise holds that $J_{T-1}(x) = K\sqrt{x}$. Here $K = \sqrt{5}$. Now if the premise holds, then

$$J_t\left(x\right) = \max_{u\in\left[0,1\right]}\left(2\sqrt{xu} + k_{t+1}\sqrt{x-ux}\right)$$

Calculating the optimal value of u gives:

$$\frac{\partial}{\partial u}\left(2\sqrt{xu} + k_{t+1}\sqrt{x-ux}\right) = \frac{x}{\sqrt{ux}} - \frac{k_{t+1}x}{2\sqrt{x-ux}} = 0$$

$$\downarrow$$

$$u = \frac{4}{4+k_{t+1}}$$

Inserting gives:

$$J_t\left(x\right) = 2\sqrt{x\frac{4}{4+k_{t+1}}} + k_{t+1}\sqrt{x - \frac{4}{4+k_{t+1}}x}$$

$$= \frac{4}{\sqrt{4+k_{t+1}}}\sqrt{x} + k_{t+1}\sqrt{\frac{k_{t+1}}{4+k_{t+1}}}\sqrt{x}$$

$$= \left(\frac{4}{\sqrt{4+k_{t+1}}} + k_{t+1}\sqrt{\frac{k_{t+1}}{4+k_{t+1}}}\right)\sqrt{x}$$