Seminar Jan 30st.

1-03

a) The easy way that requires you think before you calculate: Let $\mathbf{f} = \mathbf{a} - \mathbf{b}$, $\mathbf{g} = \mathbf{b} - \mathbf{c}$ and $\mathbf{h} = \mathbf{a} - \mathbf{c}$. Then clearly $\mathbf{f} + \mathbf{g} = \mathbf{a} - \mathbf{b} + \mathbf{b} - \mathbf{c} = \mathbf{a} - \mathbf{c} = \mathbf{h}$, so the vectors are linearly dependent. A more complicated way is to use the definition of linear independency at note that if f, g and h are linearly independent then we can find numbers x, y and z all unequal to zero such that $x(\mathbf{a} - \mathbf{b}) + y(\mathbf{b} - \mathbf{c}) + z(\mathbf{a} - \mathbf{c}) = 0$. We can rewrite the last equality as:

$$\mathbf{a}\left(x+z\right)+\mathbf{b}\left(y-x\right)-\mathbf{c}\left(y+z\right)=0$$

If the premise is true we can write this in matrix form as:

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for some vector [x y z] $\neq$ [0 0 0]. If this is the case then it must hold that:

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{vmatrix} \neq 0$$

This determinant is easily shown to be $1 + (-1) = 0$ so we can find such numbers. E.g. will [x y z] = [2.5 2.5 –2.5] do the trick.

b) Strictly speaking we know already that the vectors a – b, b – c and a – c are linearly dependent. To whit:

$$x\left(\mathbf{a}-\mathbf{b}\right)+y\left(\mathbf{b}-\mathbf{c}\right)+z\left(\mathbf{a}-\mathbf{c}\right)=4\mathbf{a}-\mathbf{b}-\mathbf{c}$$
$$\Updownarrow$$
$$x+y=4,\ \ -x+y=-1,\ \ -y-z=-1$$

The last set of equations can be written:

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

As the determinant of the 3x3 matrix has already been shown to be zero, this system has no solution or infinitely many solutions. We check the last possibility by finding the reduced row echelon form. This is easily found to be:

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which clearly is inconsistent.

1-05

We create the equation

$$\left(\mathbf{A}_1 - \lambda\mathbf{I}\right)\mathbf{x} = \left(\begin{bmatrix} 2a & 0 & 0 \\ 0 & 0 & -a \\ 2-a & 1 & 2 \end{bmatrix} - \lambda\mathbf{I}_3\right)$$

$$= \begin{bmatrix} 2a-\lambda & 0 & 0 \\ 0 & -\lambda & -a \\ 2-a & 1 & 2-\lambda \end{bmatrix}\mathbf{x} = 0$$

We then proceed to calculate what values of λ that makes the determinant $|A - \lambda I_3| = 0$.This determinant is given by:

$$\begin{aligned} \left|\mathbf{A} - \lambda\mathbf{I}_3\right| &= -\lambda\left(2a-\lambda\right)\left(2-\lambda\right) + a\left(2a-\lambda\right) \\ &= \left(2a-\lambda\right)\left(\lambda^2 - 2\lambda + a\right) \\ &= \left(2a-\lambda\right)\left(\lambda - 1 - \sqrt{1-a}\right)\left(\lambda - 1 + \sqrt{1-a}\right) \end{aligned}$$

Thus eigenvalues are given by: $\lambda_1 = 2a$, $\lambda_2 = 1 + \sqrt{1-a}$ and $\lambda_3 = 1 - \sqrt{1-a}$.

Let $a = 1$. Then the eigenvalues are : $\lambda_1 = 2$, $\lambda_2 = 1$ $\lambda_3 = 1$.

For λ_1 we can write

$$\left[\mathbf{A} - \lambda_1 \mathbf{I}_3\right]\mathbf{x} = 0$$
$$\Downarrow$$
$$\begin{bmatrix} 2-\lambda_1 & 0 & 0 \\ 0 & -\lambda_1 & -1 \\ 1 & 1 & 2-\lambda_1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 2-2 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 1 & 2-2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x} = 0$$

The last equation implies the following equations:

$$x_1 + x_2 = 0, \ -2x_2 + x_3 = 0$$

Clearly we can choose one of the variables freely. If $x_1 = t$, then $x_2 = -t$ and $x_3 = 2t$. Thus $t\times[\ 1, -1, \ 2]$ is an eigenvector. Now examine $\lambda_2 = \lambda_3 = 1$.

$$\left[\mathbf{A} - \lambda_2 \mathbf{I}_3\right]\mathbf{x} = 0$$
$$\Downarrow$$
$$\begin{bmatrix} 2-1 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}\mathbf{x} = 0$$

This equation can be written

$$x_1 = 0, \ x_2 + x_3 = 0, \ x_1 + x_2 + x_3 = 0$$

This system clearly has a solution where $x_1 = 0$ and $x_2 = -x_3$.

Thus the eigenvector is $s\times[\,0, -1, 1]$.

Now let $a = -3$. We have that

$$\left(\mathbf{A}_{-3} - \lambda\mathbf{I}\right)\mathbf{x} = \left(\begin{bmatrix} -6 & 0 & 0 \\ 0 & 0 & 3 \\ 5 & 1 & 2 \end{bmatrix} - \lambda\mathbf{I}_3\right)$$
$$= \begin{bmatrix} -6-\lambda & 0 & 0 \\ 0 & -\lambda & 3 \\ 5 & 1 & 2-\lambda \end{bmatrix}\mathbf{x} = 0$$

The determinant is given by:

$$\det\left(\mathbf{A}_{-3} - \lambda I_3\right) = \begin{vmatrix} -6-\lambda & 0 & 0 \\ 0 & -\lambda & 3 \\ 5 & 1 & 2-\lambda \end{vmatrix}$$
$$= \lambda\left(6+\lambda\right)\left(2-\lambda\right) + 3\left(6+\lambda\right)$$

Then the eigenvalues are -6, 3 and -1. For $\lambda = -6$, we have that:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 3 \\ 5 & 1 & 8 \end{bmatrix}\mathbf{x} = 0 \rightarrow \begin{array}{r} 6x_2 + 3x_3 = 0 \\ 5x_1 + x_2 + 8x_3 = 0 \end{array}$$

These equations have the solution x = $r \times$[3 1 –2].

For λ = 3, we have that:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 3 \\ 5 & 1 & -1 \end{bmatrix} \mathbf{x} = 0 \to \begin{aligned} 3x_1 &= 0 \\ -3x_2 + 3x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

These equations have the solution x = $r \times$[0 1 1].

For λ = –1, we have that:

$$\begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 3 \\ 5 & 1 & 3 \end{bmatrix} \mathbf{x} = 0 \to \begin{aligned} 3x_1 &= 0 \\ x_2 + 3x_3 &= 0 \\ 5x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

These equations have the solution x = $r \times$[0 -3 1].

1-06

(a) As the top row has the most zeros we expand along that row and get:

$$\begin{aligned}\left|D_t\right| &= t\begin{vmatrix}2 & t & 3\\ -2 & t & 0\\ 1 & 0 & 3\end{vmatrix} - \begin{vmatrix}0 & 2 & t\\ 1 & -2 & t\\ 2t & 1 & 0\end{vmatrix}\\ &= t\left(6t - 3t + 6t\right) - \left(t + 4t^2 + 4t^2\right)\\ &= t^2 - t\end{aligned}$$

Clearly if $t = 1$ or 0, we have that $|D_t|$ = zero so in this case the rank is less than 4. We now need to check if rank is 3 or less when t is 1 or 0 . If $t = 1$ we remove the first row and column and examine the determinant

$$\begin{vmatrix}2t & t & 3\\ -2 & t & 0\\ 1 & 0 & 3\end{vmatrix} = 9$$

$t = 1$ implies that $\text{rank}(D_t) = 3$. To check the rank when $t = 0$, we remove the first row and the third column. This gives the matrix

$$\begin{bmatrix}0 & 2 & 3\\ 1 & -2 & 0\\ 0 & 1 & 3\end{bmatrix}$$

The determinant of this matrix is 3 – 6 = -3, so the rank is 3 also when $t = 0$.

(b) Straightforward manipulation yields:

$$\begin{aligned}
&\mathbf{CB} + \mathbf{CXA}^{-1} = \mathbf{A}^{-1} \\
&\mathbf{C}^{-1}\mathbf{CB} + \mathbf{C}^{-1}\mathbf{CXA}^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1} \\
&\mathbf{XA}^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1} - \mathbf{B} \\
&\mathbf{X} = \mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{A} - \mathbf{BA} \\
&\mathbf{X} = \mathbf{C}^{-1} - \mathbf{BA}
\end{aligned}$$

1-07

(a) The formula for the inverse of a diagonal matrix is one those things you should now. Here is a proof. A *much* easier proof would be to use gaussian row operations. This proof uses one definition of the inverse: $A^{-1} = \det(A)^{-1} \times \operatorname{adj}(A)$. In this case is given by

$$\begin{bmatrix} \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} & \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix} & \begin{vmatrix} a & 0 \\ 0 & c \end{vmatrix} & -\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ b & 0 \end{vmatrix} & -\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \end{bmatrix} = \begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}$$

Thus

$$\mathbf{A}^{-1} = \det(\mathbf{A})^{-1} \operatorname{adj}(\mathbf{A}) = \frac{1}{abc} \begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}$$

(b) It follows directly from the definition of orthogonality.

$$\mathbf{A} = \begin{bmatrix} \mathbf{b}_1\mathbf{b}_1 & \mathbf{b}_1\mathbf{b}_2 & \mathbf{b}_1\mathbf{b}_3 \\ \mathbf{b}_2\mathbf{b}_1 & \mathbf{b}_2\mathbf{b}_2 & \mathbf{b}_2\mathbf{b}_3 \\ \mathbf{b}_3\mathbf{b}_1 & \mathbf{b}_3\mathbf{b}_2 & \mathbf{b}_3\mathbf{b}_3 \end{bmatrix}$$

If $\mathbf{b}_i$ and $\mathbf{b}_j$ are orthogonal then $\mathbf{b}_i \cdot \mathbf{b}_j = 0$. Of course, any vector with non-zero elements can not be orthogonal with itself.

(c) We have that:

$$\mathbf{A} = \mathbf{B}'\mathbf{B} \Rightarrow \mathbf{A}^{-1} = \left(\mathbf{B}'\mathbf{B}\right)^{-1} = \mathbf{B}^{-1}\mathbf{B}'^{-1}$$
$$\Downarrow$$
$$\mathbf{A}^{-1}\mathbf{B}' = \mathbf{B}^{-1}$$

(d) Orthogonality can be proven by direct calculation. Direct calculation shows that

$$\mathbf{P}'\mathbf{P} = 81\mathbf{I}_3 = \mathbf{A}$$

We then have that $A^{-1} = \frac{1}{81} I_3$. From above we have that $\mathbf{P}^{-1} = \mathbf{A}^{-1}\mathbf{P}' = \frac{1}{81}\mathbf{P}'$. As P is symmetric we can write the last bit as $81^{-1}\mathbf{P}$.

Is it so that an eigenvalue for A must be an eigenvalue for the transpose A'? (No symmetry assumption made.)

Remember (or just think) that the determinants of a matrix and its transpose are equal. Then it must hold that:

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = \det\left(\left(\mathbf{A} - \lambda\mathbf{I}\right)^{\mathbf{T}}\right) = \det\left(\mathbf{A}^{\mathbf{T}} - \lambda\mathbf{I}\right) \tag{1}$$

Let $\mathbf{A}$ be symmetric. Show that the value of the problems min/max $\mathbf{x'Ax}$ subject to $\mathbf{x'x} = 1$, is, respectively, the smallest and largest eigenvalue.

Form the Lagrangian

$$L(\mathbf{x}, \mu) = \mathbf{x'Ax} - \mu(\mathbf{x'x} - \mathbf{1})$$

Taking the derivative of the Lagrangian with respect to x gives:

$$\begin{gathered} \mathbf{Ax} + \mathbf{x'A} = \mu(\mathbf{x} + \mathbf{x'}) \\ \Downarrow \\ 2\mathbf{Ax} = 2\mu\mathbf{x} \end{gathered}$$

Clearly this implies that the value of x that solves the max/min-problem must be an eigenvector and μ must be an eigenvalue. But in that case we can write:

$$\mathbf{x'Ax} = \mathbf{x'}\mu_i\mathbf{x} = \mu_i\mathbf{x'x} = \mu_i$$

Clearly the largest eigenvalue gives the highest value and the smallest eigenvalue gives the smallest value. As $\mathbf{A}$ is symmetric we know that the eigenvalues are real. (Mathematicians need to prove existence. We can wave our hands and mutter something about the extreme value theorem.)

Let A be negative definite, let *p* be an integer and M be the power $\mathbf{A}^{p}$ of A. Show that if *p* is odd and positive, then M is negative definite def.

First start out by recalling that if $Q(\boldsymbol{x}) = \mathbf{x'Bx}$ one can always find a symmetric matrix A such that $\mathbf{x'Bx} = \mathbf{x'Ax}$ for all $x \neq 0$. Thus **B** must have the same definiteness properties as **A** and we may as well work with symmetric matrices. Recall that if a matrix **A** is symmetric, $\mathbf{A}^{-1} = (\mathbf{A}^{-1})'$.

First we check $\mathbf{xA^3x}$. Let us use the transformation $\mathbf{y_0} = \mathbf{Ax}$. Then $\mathbf{x} = \mathbf{A^{-1}y_0}$ and $\mathbf{x'} = \mathbf{y}_o\mathbf{'A^{-1}}$. We can then write $\mathbf{x'A^3x} = \mathbf{y_0'A^{-1}AAA\ A^{-1}y_0} = \mathbf{y_0'Ay_0}$ which is negative for all $\mathbf{y_0}$ if $\mathbf{x'Ax} >$ 0 for all $\mathbf{x} \neq 0$. If we then do the transformation $\mathbf{y_1} = \mathbf{Ay_0}$, we can repeat the argument for $p = 5, 7, ...$ and it should be clear that we can do so for all odd, positive p.

Decide if the same thing holds if p is even and positive.

Here we just need to come up with a counter example. Let x be one dimensional and let $\mathbf{A}$ = -1. $\mathbf{A}$ is clearly negative definite. Clearly $\mathbf{x'A}^p\mathbf{x}$ is positive when p is even so the same thing does not necessarily hold for even p.

What about negative odd powers? (That is, odd powers of the inverse of A - by the way, is it obvious that the inverse exists?)

If p is an integer and less than zero, inverses are defined as $\mathbf{A}^p = \left(\mathbf{A}^{-1}\right)^{|p|}$. We know that $\mathbf{A}^{-1}$ exist because definiteness requires full rank. And therefore $\mathbf{A}^{|p|}$ exist also for all finite p. The problem here is to show that if $\mathbf{A}$ is negative definite, then

$\mathbf{A}^{-1}$ is also negative definite. If we know this, we can use arguments from above. For instance we can do as follows. Let $\mathbf{y} = \mathbf{Ax}$. Then

$$
\begin{aligned}
\mathbf{y'A^{-1}y} &= \mathbf{y'A^{-1}Ax} \text{ (Multiply by } \mathbf{y'A}^{-1}\text{)} \\
&= \mathbf{x'AA^{-1}Ax} \quad \text{(Matrices are symmetric så } \mathbf{y'} = \mathbf{x'A'} = \mathbf{x'A}\text{)} \\
&= \mathbf{x'Ax} < \mathbf{0}
\end{aligned}
$$

Clearly if $\mathbf{A}$ is negative definite, then $\mathbf{A}^{-1}$ is also negative definite.

Exam 2009

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Consider the quadratic form

$$Q(x, y, z) = x^2 + y^2 + z^2 + 2axy + 2xz + 2yz\,.$$

(a) Prove that Q is not positive definite for any value of a.

The matrix associated with the quadratic form is:

$$\mathbf{Q} = \begin{bmatrix} 1 & a & 1 \\ a & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

In order for $\mathbf{Q}$ to be positive definit, all leading principal minors of $\mathbf{Q}$ must be positive. In particular must $|\mathbf{Q}|$ it self be strictly positive. Calculating $|\mathbf{Q}|$ gives $-a^2 + 2a - 1 \leq 0$ for all a, so positive definiteness can be ruled out.

(b) Prove that Q is positive semidefinite for one particular value of a.

If $\mathbf{Q}$ is positive semidefinite, then all principal minors must be non-negative. We already know that $|\mathbf{Q}|$ it self is never positive. Therefore, for $|\mathbf{Q}|$ to be non-negative it must be zero, which only happens if $a = 1$. But if $a = 1$, then all principal minors are either 1 or 0, so then $\mathbf{Q}$ is positive semidefinite.

(c) Put $a = 1$ and find the eigenvalues of the symmetric matrix associated with Q.

The characteristic equations for $\mathbf{Q}$ when $a = 1$ is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \lambda^2 \left(3-\lambda\right)$$

Thus $\mathbf{Q}$ in this case has an eigenvalue 3 and an eigenvalue 0 with multiplicity 2.