Seminar May 4th, Econ 4140, Eric Nævdal

Exam 2014.

Problem 1

(a) Evaluate $\int_{-\pi}^{\pi}\left(\int_{\pi}^{2\pi}\frac{\sin(xy)}{x}\,dx\right)dy$. (*Hint:* You will need a symmetry property.)

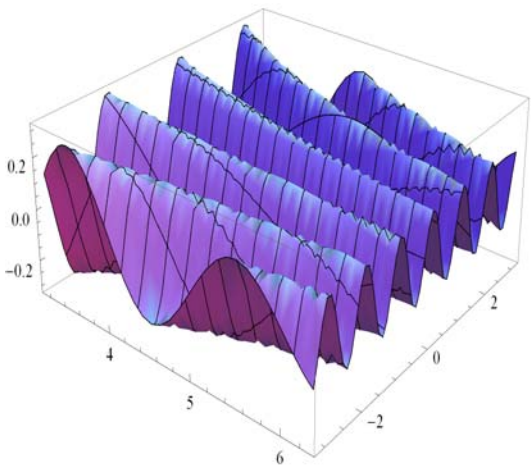
(b) Let $f(x)$ be a given C^2 strictly increasing strictly convex function defined for all real x, and define $f_1, f_2, \ldots$ inductively by

$$f_1(x) = f(x), \qquad \text{and} \qquad f_{n+1}(x) = f'(f_n(x)), \quad \text{each } n = 1, 2, \ldots$$

Use induction to show that all the f_n are quasiconvex.

(a) See solution.

Here is the function we integrate:



(b). We need two bit of information to solve this. We know that that $f(x)$ is strictly increasing and convex and therefore quasiconvex with $f'(x) > 0$ and $f''(x) > 0$. We use an inductive proof.

Steg 1: Show that $f_1(x)$ is quasiconvex. We already did this.

Steg 2: Show that if $f_n(x)$ er quasiconvex then $f_{n+1}(x) = f'(f_n(x))$ is also quasiconvex which holds if $f'(x)$ is strictly increasing which it is.

Problem 2.

Problem 2 Let $m > 0$ be a constant. Consider for each m the matrices

$$\mathbf{A} = \mathbf{A}_m = \begin{pmatrix} m^3 & \frac{3}{2}m^{-7} \\ \frac{1}{2}m^{13} & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_m = \begin{pmatrix} m^3 & \frac{3}{2}m^{-7} & 2 \\ \frac{1}{2}m^{13} & 0 & 0 \\ m^5 & m^{-5} & 4 \end{pmatrix}$$

(a) Show that $(-m^{-3}, m^7)'$ is an eigenvector for $\mathbf{A}_m$, and that its associated eigenvalue $\lambda = \lambda(m)$ is negative.

(b) Find the other eigenvalue $\mu = \mu(m)$ of $\mathbf{A}_m$, and an associated eigenvector.

(c) Find the only $m > 0$ such that $\mathbf{B}_m$ and $\mathbf{A}_m$ have same rank.

a) Vi beregner:

$$\begin{bmatrix} m^3 & \frac{3}{2}m^{-7} \\ \frac{1}{2}m^{13} & 0 \end{bmatrix}\begin{bmatrix} -m^{-3} \\ m^7 \end{bmatrix} = \lambda_1 \begin{bmatrix} -m^{-3} \\ m^7 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2}m^{10} \end{bmatrix} = \lambda_1 \begin{bmatrix} -m^{-3} \\ m^7 \end{bmatrix}$$

The last equality holds if $\lambda_1 = -\frac{1}{2}m^3$.

b)

We have that $|\mathbf{A}_m - \lambda \mathbf{I}_2| = \lambda^2 - m^3\lambda - \frac{3}{4}m = 0$. The solution is:

$$\lambda_2 = \frac{m^3 + \sqrt{m^6 - 4\frac{-3}{4}m^6}}{2} = \frac{3}{2}m^3$$

In order to find the eigenvector associated with λ_2, we must have:

$$\begin{bmatrix} m^3 & \frac{3}{2}m^{-7} \\ \frac{1}{2}m^{13} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{3}{2}m^3\begin{bmatrix} x \\ y \end{bmatrix}$$

Written on scalar form we have two equations, but we only need one of them. E.g. we have that $2m^3x + 3m^{-7}y = 3m^3x$. This implies that $3y = m^{10}x$ so the eigenvector is given by:

$$\mathbf{v} = r\begin{bmatrix} 1 \\ \frac{1}{3}m^{10} \end{bmatrix}$$

Here r is an arbitrary non-zero scalar.

c) Rank($\mathbf{A}_m$) is obviously 2. Rank($\mathbf{B}_m$) $\leq$ 3. If det($\mathbf{B}_m$) $= 0$, Rank($\mathbf{B}_m$) $=$ Rank($\mathbf{A}_m$). det($\mathbf{B}_m$) $= m^8 - 3m^6 = m^6(m^2 - 3) = 0$ $\rightarrow m = \sqrt{3}$.

Problem 3)

Problem 3 Let G and H be C^2 functions defined on $(0,\infty)$, let $m > 0$ be a constant and S be the open first quadrant $S = \{(x,y);\ x > 0,\ y > 0\}$. For $x = x(t)$, $y = y(t)$, consider the differential equation system – valid from time $t = 0$ until the first time $T \geq 0$ for which $(x(T), y(T)) \notin S$:

$$\begin{aligned} \dot{x} &= G(x) + H(y) \\ \dot{y} &= \left[m^3 - G'(x)\right] \cdot y \end{aligned} \qquad \text{(D)}$$

(Observe that there is a derivative sign «G'» in the second equation.)

(a) Show that if $H' > 0 > G''$ (so that in particular, $m^3 - G'$ is strictly increasing), then

(i) the system has at most one equilibrium point in S (note $xy > 0$ in S!), and

(ii) if such one exists, it is a saddle point. (*Hint:* a term will vanish and simplify.)

Let from now on $G(x) = 2x^{1/2}$ and $H(y) = -2y^{-3/4}$ so that the saddle point has coordinates $(\bar{x}, \bar{y}) = (m^{-6}, m^4)$. (You need not show this.)

(b) Put $m = 1$. For those two integral curves (i.e. particular solution trajectories) $(x(t), y(t))$ which converge to $(\bar{x}, \bar{y})$ as $t \to +\infty$, show that the *slope* $\frac{y(t)-\bar{y}}{x(t)-\bar{x}}$ converges to -1. (*Hint:* Problem 2 gives information which likely saves time.)

(c) Put $m = 1$. Sketch a phase diagram and indicate a few representative integral curves.

a)

(i) If there is an equilibrium in S, it must solve $\dot{y} = 0 \rightarrow m^3 - G'(x) = 0$. As $m^3 - G'(x)$ is strictly increasing there is at most one value of x, denoted x_{ss} that solves the equation. We then must find a solution to $G(x_{ss}) = -H(y)$. Again, as $H(y)$ is strictly increasing this equation has at most one solution, denoted y_{ss}.

(ii) We linearize the system around the equilibrium point (x_{ss}, y_{ss}). This yields

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} G'\left(x_{ss}\right) & H'\left(y_{ss}\right) \\ -G''\left(x_{ss}\right) & \left(m^3 - G'\left(x_{ss}\right)\right) \end{bmatrix} \begin{bmatrix} x - x_{ss} \\ y - y_{ss} \end{bmatrix}$$
$$= \begin{bmatrix} G'\left(x_{ss}\right) & H'\left(y_{ss}\right) \\ -G''\left(x_{ss}\right) & 0 \end{bmatrix} \begin{bmatrix} x - x_{ss} \\ y - y_{ss} \end{bmatrix}$$

We can calculate that the determinant of the 2×2 matrix is $G''\left(x_{ss}\right)H'\left(y_{ss}\right)$. As $G'' < 0$ and $H'(y) > 0$, this determinant is negative so the system is a saddle point.

b) We now denote x_{ss} as $\overline{x}$ and y_{ss} as $\overline{y}$. For $m = 1$, the system is then:

$$\dot{x} = 2\sqrt{x} - \frac{2}{y^{3/4}}, \ \dot{y} = \left(1 - \frac{1}{\sqrt{x}}\right)y$$

We linearise the system around its steady state and get the following system on matrix form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{x_{ss}}} & \dfrac{3}{2y_{ss}^{7/4}} \\ \dfrac{yss}{2x_{ss}^{3/2}} & 1 - \dfrac{1}{\sqrt{x_{ss}}} \end{pmatrix} \begin{bmatrix} x - x_{ss} \\ y - y_{ss} \end{bmatrix}$$

$$= \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{bmatrix} x - x_{ss} \\ y - y_{ss} \end{bmatrix}$$

We recognize the matrix in the last expression as $\mathbf{A}_1$ from the previous exercise. In 2a) we have been informed that the eigenvector associated with the negative eigenvalue is [–1, 1]. Therefore the slope of the stable saddle path is –1 in $(x_{ss}, y_{ss}) =$ (1, 1).

Problem 4.

Problem 4 Let $x_0 > 0$ and consider – but do not solve! – the optimal control problem

$$V(x_0) = \max_{u(t)\geq 0} \int_0^{2014} \frac{-6e^{-t}}{u(t)}\,dt, \quad \text{where} \quad x(0) = x_0, \quad x(2014) \geq 0, \qquad \dot{x} = 2x^{1/2} - 2u^3.$$

(a) State the conditions from the maximum principle.
(You can safely disregard the «p_0» constant and put it $= 1$).

(b) Let $x(t)$ satisfy the conditions from the maximum principle with adjoint variable $p(t)$.
Let $y(t) = e^t p(t)$, so that $\dot{y} = y + e^t \dot{p}$ (then y is the current-value adjoint).
Show that (x, y) satisfies the differential equation system (D) of Problem 3, with $G(x) = 2x^{1/2}$, $H(y) = -2y^{-3/4}$ and $m = 1$ (as in Problem 3 part (c)).
(*Hint:* you shall obtain the condition $u(t) = \big(y(t)\big)^{-1/4}$.)

(c) *«Bonus» question: this part will be deleted (zero-weighted) if that benefits your grade.*
Consider your phase diagram for Problem 3 part (c), and assume $x(0) = x_0 = 1 = \bar{x}$ (the x-coordinate of the saddle point). Take for granted that the optimal path x^* ends at $x^*(2014) = 0$. Use this to argue for an upper *or* a lower bound for $V'(1)$; i.e.,

Find an appropriate $a > 0$ and
- either argue that $V'(1) \leq a$
- or argue that $V'(1) \geq a$.

(Recall that $V = V(x_0)$ is the optimal value as function of initial state x_0.)

(a) The problem is:

$$\max_{u(t)\geq 0} \int_0^{2014} \frac{-6e^{-t}}{u}\,dt \quad s.t: x(0) = x_0,\ x(2014) \geq 0,\ \dot{x} = 2\sqrt{x} - 2u^3$$

We form the Hamiltonian:

$$H = \frac{-6e^{-t}}{u} + p\left(2\sqrt{x} - 2u^3\right)$$

We note that H is strictly concave in x and as

$$\partial^2 H \Big/ \partial u^2 = -\frac{12e^{-t}}{u^3} - 12pu < 0$$

the Hamiltonian is clearly concave in u as well as long as p is positive. The maximum principle gives the following conditions:

$$u = \max\left(0, \frac{e^{-t/4}}{p^{1/4}}\right)$$
$$\dot{p} = -\frac{p}{\sqrt{x}}$$
$$\dot{x} = 2\sqrt{x} - 2u^3$$

In addition we have the transversality condition $p(2014) \geq 0$, $(= 0$ if $x(2014) > 0)$.

(b). If $p = ye^{-t}$ then u may be written:

$$u = \max\left(0, \frac{e^{-t/4}}{\left(ye^{-t}\right)^{1/4}}\right) = \max\left(0, \frac{1}{y^{1/4}}\right) = y^{-1/4}$$

Thus we can write $\dot{x} = 2\sqrt{x} - 2y^{-3/4}$. As $y = y + e^t \dot{p}$ we have that $\dot{p} = \left(\dot{y} - y\right)e^{-t}$. This implies that we can write

$$\dot{p} = \left(\dot{y} - y\right)e^{-t} = -\frac{\partial H}{\partial x} = \frac{p}{\sqrt{x}} = \frac{ye^{-t}}{\sqrt{x}}$$

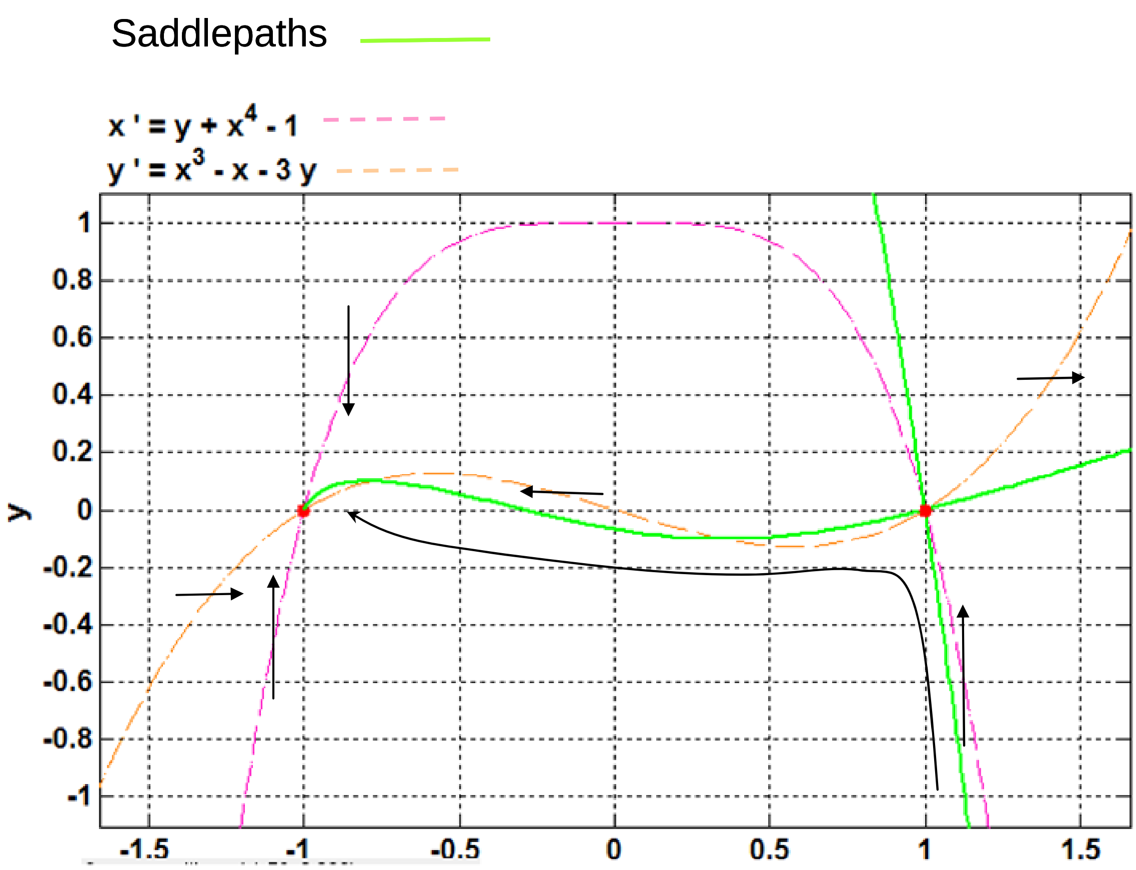
$$\Downarrow$$

$$\dot{y} = y - \frac{y}{\sqrt{x}} = \left(1 - \frac{1}{\sqrt{x}}\right)y$$

Thus they are the same.

Exam 2017

1.c) The phase diagram



Problem 3b)

When $m = 0$, then the maximum principle states that:

$$\frac{\partial H}{\partial u} = -x^{k+1} + px > 0 \rightarrow "u = \infty"$$
$$\frac{\partial H}{\partial u} = -x^{k+1} + px < 0 \rightarrow u = 0$$
$$\dot{p} = -2x + u\left(k+1\right)x^{k} - pu$$

We try the solution $u = 0$. Then $x = x_0$ and $p =$ -$2x_0(t - T)$.

Bur then we must have then we must have that:

$$\frac{\partial H}{\partial u} = -x_0^{k+1} - 2x_0^2\left(t - T\right) < 0 \quad \text{for all } t \geq 0$$

Since this expression is increasing in t, we now that if it holds for $t = 0$, it holds for all t. Therefore:

$$\frac{\partial H}{\partial u} = -x_0^{k+1} + 2x_0^2 T < 0 \rightarrow T < \frac{1}{2}x_0^{k-1}$$