Answers to DIY questions, Lecture 3

Ragnar Nymoen

January 24, 2012

DYI Exercise 1

Consistency extends to $\hat{\beta}_2$ and $\hat{\alpha}$ since we have $E(\hat{\beta}_2) = \beta_2$ and $E(\hat{\alpha}) = \alpha$ and the expressions for $var(\hat{\beta}_2)$ and $var(\hat{\alpha})$ show that both approach zero as n grows towards infinity.

DYI Exercise 2

$$\hat{e}_i = y_i - \widehat{\beta}_1 - \widehat{\beta}_2 x_i \equiv y_i - \widehat{\alpha} - \widehat{\beta}_2(x_i - \bar{x})$$

since

$$\widehat{\alpha} = \widehat{\beta}_1 + \widehat{\beta}_2 \bar{x}$$

Then

$$\begin{aligned} \sum_{i=1}^{n} \hat{e}_i &= \sum_{i=1}^{n} \left(y_i - \widehat{\alpha} - \widehat{\beta}_2(x_i - \bar{x}) \right) = \\ &= \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_i - \widehat{\beta}_2 \sum_{i=1}^{n} (x_i - \bar{x}) = 0 \end{aligned}$$

DYI Exercise 3

The normal equations:

$$\begin{aligned} \bar{y} - \hat{\alpha} &= 0 \\ \sum_{i=1}^{n} \{(x_i - \bar{x})y_i - \widehat{\beta}_2 (x_i - \bar{x})^2\} &= 0 \end{aligned}$$

The second equation can be written as:

$$\begin{aligned} \sum_{i=1}^{n} \{y_i - \hat{\alpha} - \widehat{\beta}_2(x_i - \bar{x}) + \hat{\alpha}\}(x_i - \bar{x}) &= 0 \Longrightarrow \\ \sum_{i=1}^{n} \hat{e}_i (x_i - \bar{x}) + \hat{\alpha} \sum_{i=1}^{n} (x_i - \bar{x}) &= 0 \Longrightarrow \end{aligned}$$

$$\sum_{i=1}^{n} \hat{e}_i (x_i - \bar{x}) = 0 \tag{1}$$

DYI Exercise 4

OLS applied to

$$y_i = \beta_2 x_i + e_i' \tag{2}$$

means choosing $\hat{\beta}_1'$ so that

$$S'(\hat{\beta}_2') = \sum_{i=1}^{n}(y_i - \hat{\beta}_2' x_i)^2$$

is minimized. This gives the 1 o c:

$$\sum_{i=1}^{n} y_i x_i - \hat{\beta}_2' \sum_{i=1}^{n} x_i^2 = 0$$

so that the OLS estimator when we estimate (2) is

$$\hat{\beta}_2' = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \tag{3}$$

and the corresponding residuals (from the regression without an intercept):

$$\hat{e}_i' = y_i - \hat{\beta}_2' x_i, \ \ i = 1, 2, \ldots, n$$

From the OLS principle we only get

$$\begin{aligned} \sum_{i=1}^{n}(y_i - \hat{\beta}_2' x_i)x_i &= 0 \text{ (1 oc for minimum)} \\ \sum_{i=1}^{n} \hat{e}_i' x_i &= 0 \end{aligned}$$

which does not imply $\sum_{i=1}^{n} \hat{e}_i' = 0$, unless:

1. $x_i =$ a *constant* for all i. But then we are estimating the constant mean of y_i rather than the slope coefficient, or
2. Both y and x have zero means, so that $\bar{y} = \bar{x} = 0, \Longrightarrow \hat{\beta}_2' = \hat{\beta}_2$.

Conclusion: In general OLS estimation of β_2 in an equation that omits the interept term, leads to residuals that do not sum to 0:

$$\sum_{i=1}^{n} \hat{e}_i' \neq 0$$

DYI Exercise 5

If the true model is

$$y_i = \beta_1 + \beta_2 x_i + e_i, \, \beta_1 \neq 0 \tag{4}$$

with disturbance properties as in RM1, then

$$\begin{aligned} \hat{\beta}_2' &= \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i \{\beta_1 + \beta_2 x_i + e_i\}}{\sum_{i=1}^{n} x_i^2} \\ &= \beta_2 + \beta_1 \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2} + \frac{\sum_{i=1}^{n} e_i x_i}{\sum_{i=1}^{n} x_i^2} \end{aligned}$$

with

$$E(\hat{\beta}_2') = \beta_2 + \beta_1 \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2} \neq \beta_2$$

meaning that the estimator $\hat{\beta}_2'$ is biased when the true model is (4).