Expectations models, measurement-error, Lucas critique, invariance (Lect 17)

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References

- HGL, Ch 9.7.2 (forecasting with an ARDL model), and Ch. 10.2. (Measurement error).
- BN Kap 5.11-5.12

Recap: OLS and systems-of-equations estimation

- Lessons from Lecture 16:
- For the estimation of parameters of a system of equation OLS may, or may not, be used to obtain consistent estimators of the parameters of interest.
- If the *parameters of interest* are the parameters of one or more of the structural equations of the model, OLS will not give consistent estimators
- This is known as the "simultaneous-equations bias" or "simultaneity bias".
- If the *parameters of interest* are the parameters of (one or more) of the reduced form equations, OLS give consistent estimators.
- If the purpose of the analysis is forecasting, the relevant parameters of interest are of the reduced forms equations (in our example for C_t) so OLS can be used.

Subjective and rational expectations I

- So far, agents' expectations have not been explicit in the models we have considered.

$$Y_t = \beta_0 + \beta_1 X_t + \varepsilon_t \tag{1}$$

Tentative general interpretation: Agents formulate contingent plans, and act on observed X in period t.

- Formulation with explicit expectation notation:

$$Y_t = \beta_0 + \beta_1 X_t^e + \varepsilon_t, \tag{2}$$

where X_t^e represents the agents' expected current variable.

Subjective and rational expectations II

- In modern economic theory it is common with models which contain *leads* of expectations variables:

$$Y_t = \beta_0 + \beta_1 X^e_{t+1} + \varepsilon_t. \qquad (3)$$

- To complete the econometric model, need to specify how expectations are generated (formed). There are two main hypotheses:
- **Subjective expectations:** X^e_{t+1} based on observations of the random variable up to $t-1$ (or t).
- **Rational expectation**: X^e_{t+1} is defined as the *mathematical conditional expectation* of X^e_{t+1} based on a information set $\mathcal{I}_{t-1}$ or $(\mathcal{I}_t)$.

Subjective expectations I

- Example (a "classic"): Adaptive expectations:

$$X^e_{t+1} = (1-\tau)X_t + \tau X^e_t,\ 0 < \tau < 1 \tag{4}$$

Solution for X^e_{t+1}:

$$X^e_{t+1} = (1-\tau)\sum_{j=0}^{\infty}\tau^j X_{t-j} \tag{5}$$

Insertion in (3) gives

$$Y_t = \beta_0 + \beta_1(1-\tau)\sum_{j=0}^{\infty}\tau^j X_{t-j} + \varepsilon_t \tag{6}$$

Subjective expectations II

Use of the "Koyck-transformation" from Lecture 15 gives:

$$Y_t = \beta_0 - \tau\beta_0 + \tau Y_{t-1} + \beta_1(1-\tau)X_t + \varepsilon_t - \tau\varepsilon_{t-1}$$
$$Y_t = \beta_0^* + \beta_1^* Y_{t-1} + \beta_2^* X_t + \varepsilon_t - \tau\varepsilon_{t-1} \quad (7)$$

- Assume that conditional on Y_{t-1} and X_t, ε_t $(t = 1, 2, ...)$ have classical properties.
- How would you estimate β_0^*,β_1^*, and β_2^* in (7)?

Rational Expectations (RE) I

RE theory and econometrics motivated the "Nobel Price in Economics" to Sargent and Sims in 2011.

> *"...it emphasized rational expectations, the notion that economic decisions makers like households and firms do not make systematic mistakes in forecasting" The Royal Swedish Academy of Sciences, 2011*

An example RE model:

$$Y_t = \beta_0 + \beta_1 E(X_{t+1} \mid \mathcal{I}_{t-1}) + \varepsilon_t \quad (8)$$

$$X_t = \lambda X_{t-1} + \epsilon_{xt}, \quad -1 < \lambda < 1 \quad (9)$$

where ε_t and ϵ_{xt} are two disturbances that are independent from each other, $Cov(\varepsilon_t, \epsilon_{xt}) = 0$, and they have classical properties conditional on the agents' information set $\mathcal{I}_{t-1}$.

Rational Expectations (RE) II

- In (8)-(9) the only information about $t-1$ is X_{t-1}, so conditioning on $\mathcal{I}_{t-1}$ is the same as conditioning on X_te.
- What is $E(X_{t+1} \mid \mathcal{I}_{t-1}) \equiv E(X_{t+1} \mid \mathcal{I}_{t-1})$?
- The answer is obtained from the reduced form of X_{t+1} because that equation contains all information about the random variable X_{t+1}:

$$\begin{aligned} X_{t+1} &= \lambda X_t + \epsilon_{t+1} \\ &= \lambda^2 X_{t-1} + \lambda \epsilon_{xt} + \epsilon_{xt+1}, \end{aligned} \tag{10}$$

Then

$$E(X_{t+1} \mid X_{t-1}) = \lambda^2 X_{t-1} \tag{11}$$

Rational Expectations (RE) III

Replacing the expected X_{t+1} in (8) by rhs of (11) gives

$$\begin{aligned} Y_t &= \beta_0 + \beta_1 \lambda^2 X_{t-1} + \varepsilon_t \\ &= \beta_0 + \beta_1^{'} X_{t-1} + \varepsilon_t \end{aligned} \tag{12}$$

which is the r*ational expectations solution* for Y_t.

- The parameters of (12) are estimated consistently by OLS, Why?
- But the parameter of interest, β_1, is not estimated consistently by that OLS estimator:

$$plim(\hat{\beta}_1^{'}) = plim(\frac{\sum_t (X_{t-1} - \bar{X}_{-1}) Y_t}{\sum_t (X_{t-1} - \bar{X}_{-1})^2}) = \beta_1 \lambda^2 \neq \beta_1$$

Rational expectations and bias of OLS I

From

$$E(X_{t+1} \mid X_{t-1}) = \lambda^2 X_{t-1}$$

and the reduced form (10), we have that

$$X_{t+1} - E(X_{t+1} \mid X_{t-1}) = \lambda \epsilon_{xt} + \epsilon_{xt+1}$$

and

$$E(X_{t+1} \mid X_{t-1}) = X_{t+1} - \lambda \epsilon_{xt} - \epsilon_{xt+1}$$

Rational expectations and bias of OLS II

- The forecast error has zero mean, and has constant variance $(\lambda^2 + 1)\sigma_x^2$. This makes it tempting to $E(X_{t+1} \mid \mathcal{I}_{t-1})$ in

$$Y_t = \beta_0 + \beta_1 E(X_{t+1} \mid \mathcal{I}_{t-1}) + \varepsilon_t$$

by X_{t+1} and use OLS to estimate

$$Y_t = \beta_0 + \beta_1 X_{t+1} + u_t \tag{13}$$

and "hope for the best".

- However, since

$$u_t = \varepsilon_t - \beta_1 \lambda \epsilon_{xt} - \beta_1 \epsilon_{xt+1}$$

there is no way that we can claim independence between the disturbance u_t and the explanatory variable X_{t+1}.

- The OLS estimator $\hat{\beta}_1$ from (13) is therefore inconsistent for the parameter of interest β_1 in the model (8)-(9).

A closer look at the OLS bias in RE models I

Consider the even simpler RE model:

$$Y_t = \beta_1 E(X_t \mid \mathcal{I}_{t-1}) + \varepsilon_t \tag{14}$$

$$X_t = \lambda X_{t-1} + \epsilon_{xt}, \quad -1 < \lambda < 1 \tag{15}$$

with the same assumptions about ε_t and ε_{Xt} as above.
We now have:

$$X_t = E(X_t \mid \mathcal{I}_{t-1}) + \epsilon_{xt} = \lambda X_{t-1} + \epsilon_{xt} \tag{16}$$

and if we replace $E(X_t \mid \mathcal{I}_{t-1})$ by X_t in (14), the disturbance u_t in

$$Y_t = \beta_1 X_t + u_t$$

A closer look at the OLS bias in RE models II

must be

$$u_t = \varepsilon_t - \beta_1 \epsilon_{xt}.$$

The probability limit of the OLS estimator

$$\hat{\beta}_1 = \frac{\sum_{t=1}^{T} Y_t X_t}{\sum_{t=1}^{T} X_t^2}$$

$$\begin{aligned} plim(\hat{\beta}_1 - \beta_1) &= \frac{plim \frac{1}{T} \sum_{t=1}^{T} u_t X_t}{plim \frac{1}{T} \sum_{t=1}^{T} X_t^2} \\ &= \frac{plim \frac{1}{T} \sum_{t=1} \left(\varepsilon_t - \beta_1 \epsilon_{xt}\right)\left(\lambda X_{t-1} + \epsilon_{xt}\right)}{\frac{\sigma^2_{\epsilon_{xt}}}{1-\lambda^2}} \\ &= \frac{-\beta_1 \sigma^2_{\epsilon_x}}{\frac{\sigma^2_{\epsilon_x}}{1-\lambda^2}} = -\beta_1 (1 - \lambda^2) \qquad (17) \end{aligned}$$

Measurement-error bias I

- The bias of OLS in RE model is a special case of the phenomenon called *measurement-error* bias.
- To see the generality of the problem, consider the cross-section model:

$$Y_i = \beta_0 + \beta_1 X_i^* + \varepsilon_i^* \;\; i = 1, 2, \ldots, n \tag{18}$$

where $Cov(\varepsilon_i^*, X_i^*) = 0$, and ε_i^* has the other classical properties as well.

- Assume that X_i^* is an unobservable random variable which is replaced by the observable X_i. in the estimation of (18). The difference between X_i and X_i^* is random:

$$e_i = X_i - X_i^*$$

Measurement-error bias II

- Even if all e_i and ε_i^* are independent, OLS on

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \;\; i = 1, 2, \ldots, n \tag{19}$$

will produce an inconsistent estimator of β_1, because $Cov(\varepsilon_i, X_i) \neq 0$. The bias can be shown (try it) to be

$$plim(\hat{\beta}_1 - \beta_1) = \frac{-\beta_1 \sigma_e^2}{Var(X)} = \frac{-\beta_1 \sigma_e^2}{\sigma_{X^*}^2 + \sigma_e^2} \tag{20}$$

where $\sigma_e^2 = Var(e_i)$ and $\sigma_{X^*} = Var(X_i^*)$.

- The RE bias in (17) is a special case (adjusting for notation) of (20). See sepeate lecture note (posted 6 May)

The Lucas critique I

- Return to the time series case, and the RE model (14)-(14). We know that the OLS estimator is inconsistent:

$$plim(\hat{\beta}_1) = \beta_1 - \beta_1(1 - \lambda^2) = \beta_1\lambda^2 \qquad (21)$$

- Note that $\beta_1\lambda^2$ is the slope coefficient of the conditional expectation of Y_t given X_t.
- Why? Because OLS is **always** a consistent estimator of the conditional expectation.

The **Lucas-critique,** from 1976, attacks the idea that if there is a change in the expectations about X_t, the effect of this change on Y_t can be predicted by using the OLS estimate $\hat{\beta}_1$ from a regression model.

The Lucas critique II

- The critique says that if the true model is a RE model of the type (14)-(14), then $plim(\hat{\beta}_1)$ has to change in the same time period as expectations change, i.e. when λ changes.
- The critique implies that policy analysis cannot be based on OLS estimated conditional expectations (regression models).

Testing the relevance of the Lucas critique I

- The Lucas-critique is a "possibility theorem" not a truism
- If there is evidence of a structural breaks in the equation for X_t we can test the relevance of the Lucas critique.
- Logically, the combined occurrence of
  - Structural breaks in the equation for X_t
  - Structural breaks in the parameters of the conditional model for Y_t given X_t (i.e. the regression model)

  confirm the Lucas critique
- But conversely, the combined occurrence of
  - Structural breaks in the equation for X_t
  - Invariance (no break) in the conditional model

refutes the relevance of the Lucas critique.

Testing the relevance of the Lucas critique II

- The case where the parameters of the conditional model are invariant to structural breaks elsewhere in the system, is called the case of super-exogenous regressors
- Invariance is a relative property: No model can have parameters that are invariant to *all* types of shocks, and we can only test for the ones that have occurred.
- Haavelmo was clear about this already in 1944, in *The Probability Approach in Econometrics*. He called equations with parameters that have a high degreee of invariance to changes elsewhere in the system *autonomous equations*. He also said that:
  The construction of systems of autonomous equations is a matter of intuition and factual knowledge, it is an art (p 29)

A simple analysis of invariance I

- Consider a sample with time series data for X_t and Y_t $(t = 1, 2, \ldots T)$
- Start by recording the two possible regression coefficients:
  - Regress Y on X :

$$\hat{\beta}_1 = \frac{\sum_{t=1}^{T}(X_t - \bar{X})Y_t}{\sum_{t=1}^{T}(X_t - \bar{X})^2}$$

  - Regress X on Y:

$$\hat{\beta}_1' = \frac{\sum_{t=1}^{T}(X_t - \bar{X})Y_t}{\sum_{t=1}^{n}(Y_t - \bar{Y})^2}$$

A simple analysis of invariance II

- As we have seen many times

$$\hat{\beta}_1\hat{\beta}_1^{'} = r^2_{XY} \tag{22}$$

and

$$plim(r^2_{XY}) = \rho^2_{XY} \text{ (theoretical correlation)}$$
$$\beta_1\beta_1^{'} = \rho^2_{XY}$$

- Assume that there is a change to X_t that leads to a break in ρ_{XY}. The Lucas critique says that in particular $\beta_1$should change at the same point in time.
- If $\hat{\beta}_1$ remains constant emprically despite the break in ρ_{XY}, the relevance of the Lucas critique is refuted, there is a degree of invariance in $plim(\hat{\beta}_1)$.

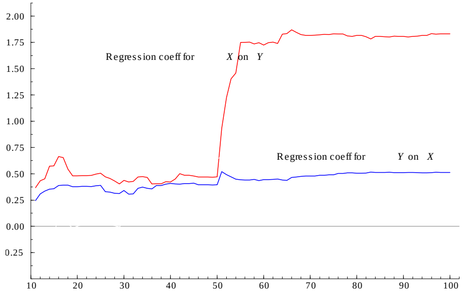
A simple analysis of invariance III

- This can be tested by a regression model with an interaction term

$$Y_t = \beta_0 + \beta_1 X_t + \delta X_t D_t + \varepsilon_t$$

  where the dummy D_t captures the break. A step-dummy if there is a permanent break. Test H_0 $\delta = 0$ by the t-ratio test
- Or can use graphs of recursive estimates, as in Lecture 1:

An example of super-exogeneity I



- The graphs show recursive estimates of $\hat{\beta}_1$and $\hat{\beta}_1'$
- There is a structural break in period 50 (a higher σ_X that reduces $plim(r_{XY})$, and λ on the previous slides)
- $\hat{\beta}_1$ is invariant, refuting the Lucas critique and supporting that $X \to Y$.
- More about invariance and super-exogeneity and expectations in E 4160.