Lecture note about $cov(\hat{\alpha},\hat{\beta}_1)$ to accompany Lecture 4 slide set

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This note is a translation of Appendix 3.A in BN. We include it as documentation and for completeness. If you are interested in this kind of exercise and can formulate a more elegant proof, let me know!

With reference to the notation in Lecture 4 we have

$$
\begin{aligned}
Cov\left(\hat{\alpha},\hat{\beta}_1\right) &= E\left[(\hat{\alpha}-\alpha)\left(\hat{\beta}_1-\beta_1\right)\right] = E\left[\hat{\alpha}\left(\hat{\beta}_1-\beta_1\right)-\alpha\left(\hat{\beta}_1-\beta_1\right)\right] \\
&= E\left[\hat{\alpha}\left(\hat{\beta}_1-\beta_1\right)\right] \qquad (1)
\end{aligned}
$$

and we want to show that $E\left[\hat{\alpha}\left(\hat{\beta}_1-\beta_1\right)\right]=0$.

Start but noting that $\hat{\beta}_1-\beta_1$:

$$
\begin{aligned}
\hat{\beta}_1-\beta_1 &= \hat{\beta}_1 - E\left(\hat{\beta}_1\right) = \qquad (2) \\
&= \frac{\sum_{i=1}^{n} Y_i\left(X_i-\bar{X}\right)}{n\hat{\sigma}_x^2} - \frac{\sum_{i=1}^{n} E\left(Y_i\right)\left(X_i-\bar{X}\right)}{n\hat{\sigma}_x^2} \\
&= \frac{1}{n\hat{\sigma}_x^2}\sum_{i=1}^{n}\left[Y_i-E\left(Y_i\right)\right]\left(X_i-\bar{X}\right) \\
&= \frac{1}{n\hat{\sigma}_x^2}\sum_{i=1}^{n}\varepsilon_i\left(X_i-\bar{X}\right),
\end{aligned}
$$

where we have used that

$$
Y_i - E\left(Y_i\right) = Y_i - \alpha - \beta_1\left(X_i-\bar{X}\right) = \varepsilon_i.
$$

Next, use the expression $\hat{\beta}_1-\beta_1$ in the definition of $cov\left(\hat{\alpha},\hat{\beta}_1\right)$ in (1):

$$
\begin{aligned}
E\left[\hat{\alpha}\left(\hat{\beta}_1-\beta_1\right)\right] &= E\left[\frac{1}{n}\sum_{j=1}^{n}Y_j\frac{1}{n\hat{\sigma}_x^2}\sum_{i=1}^{n}\varepsilon_i\left(X_i-\bar{X}\right)\right] \\
&= \frac{1}{n^2\hat{\sigma}_x^2}E\left[\sum_{j=1}^{n}Y_j\sum_{i=1}^{n}\varepsilon_i\left(X_i-\bar{X}\right)\right],
\end{aligned}
$$

where we have used $\hat{\alpha} = \bar{Y}$.

Consider the case of $n = 2$: By inspection, the expression after the second equality sign becomes

$$\frac{1}{4\hat{\sigma}_x^2} E\left[\sum_{j=1}^{2} Y_j \sum_{i=1}^{2} \varepsilon_i \left(X_i - \bar{X}\right)\right]$$
$$= \frac{1}{4\hat{\sigma}_x^2}\left\{E\left[Y_1\varepsilon_1\left(X_1-\bar{X}\right) + Y_1\varepsilon_2\left(X_2-\bar{X}\right) + Y_2\varepsilon_1\left(X_1-\bar{X}\right) + Y_2\varepsilon_2\left(X_2-\bar{X}\right)\right]\right\},$$

i.e., the sum of all cross products between Y_j and $\varepsilon_i(X_i - \bar{X})$. A typical term in $\sum_{j=1}^{n} Y_j \sum_{i=1}^{n} \varepsilon_i \left(X_i - \bar{X}\right)$ is

$$\begin{aligned} E[Y_j\varepsilon_i\left(X_i-\bar{X}\right)] &= E\left\{\left[\alpha + \beta_1\left(X_j - \bar{X}\right) + \varepsilon_j\right]\varepsilon_i\left(X_i - \bar{X}\right)\right\} \\ &= E\left[\varepsilon_j\varepsilon_i\left(X_i-\bar{X}\right)\right] \\ &= \begin{cases} 0 & \text{when } i \neq j \\ \sigma^2\left(X_i - \bar{X}\right), & \text{when } i = j \ (n \text{ times}), \end{cases} \end{aligned}$$

By this argument, we see that the expression for $Cov\left(\hat{\alpha}, \hat{\beta}_1\right)$ simplifies to

$$Cov\left(\hat{\alpha}, \hat{\beta}_1\right) = \frac{\sigma^2}{n^2\hat{\sigma}_x^2}\sum_{i=1}^{n}\left(X_i - \bar{X}\right) = 0. \tag{3}$$