Answer to "Lecture question" in Lecture 3

In Lecture 3 under the heading **Final equation**, we studied the Gaussian VAR

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix}, \tag{1}$$

where $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the matrix of autoregressive coefficients and we assume that

$$\begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix} \sim IID\left(\mathbf{0}, \begin{matrix} \sigma_y^2 & \sigma_{yx} \\ \sigma_{yx} & \sigma_x^2 \end{matrix}\right) \forall\ t \tag{2}$$

As an exercise, you were asked to show that (1) can be reduced to the so called **final equation** for Y_{t+1}

$$Y_{t+1} = (\underbrace{a_{11} + a_{22}}_{\equiv \rho_1})Y_t + (\underbrace{a_{12}a_{21} - a_{22}a_{11}}_{\equiv \rho_2})Y_t + \underbrace{\varepsilon_{yt+1} - a_{22}\varepsilon_{yt} + a_{12}\varepsilon_{xt}}_{\equiv \varepsilon_t}. \tag{3}$$

Answer:

Solve the equation in the first row in (1) for X_{t-1}:

$$X_{t-1} = (1/a_{12})Y_t - (a_{11}/a_{12})Y_{t-1} - (1/a_{12})\varepsilon_{yt}. \tag{4}$$

Substitution in the second row of (1) gives

$$X_t = \frac{a_{22}}{a_{12}}Y_t + (a_{21} - a_{22}\frac{a_{11}}{a_{12}})Y_{t-1} - \frac{a_{22}}{a_{12}}\varepsilon_{yt} + \varepsilon_{xt},\ a_{12} \neq 0. \tag{5}$$

Finally: Change t to $t+1$ in the first row of (1), and replace X_t by the expression on the right hand side of (5):

$$Y_{t+1} = a_{11}Y_t + a_{12}\left(\frac{a_{22}}{a_{12}}Y_t + (a_{21} - a_{22}\frac{a_{11}}{a_{12}})Y_{t-1} - \frac{a_{22}}{a_{12}}\varepsilon_{yt} + \varepsilon_{xt}\right) + \varepsilon_{yt+1} \tag{6}$$

Collecting terms gives

$$Y_{t+1} = (a_{11} + a_{22})Y_t + (a_{12}a_{21} - a_{22}a_{11})Y_{t-1} + \varepsilon_{yt+1} - a_{22}\varepsilon_{yt} + a_{12}\varepsilon_{xt}, \tag{7}$$

which is what we should find. Clearly this equation must hold for all periods, so we can write

$$Y_t = (a_{11} + a_{22})Y_{t-1} + (a_{12}a_{21} - a_{22}a_{11})Y_{t-2} + \varepsilon_{yt} - a_{22}\varepsilon_{yt-1} + a_{12}\varepsilon_{xt-1}, \tag{8}$$

the **final equation** for Y_t defined by the system (1).

A final equation expresses an endogenous variable by the "its own lags" and exogenous random variables. No lags of other endogenous variables are allowed in a final equation expression.