ECON 4160, Spring term 2013. Lecture 9

SVAR and recursive models. Encompassing. SUR systems

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- References to Davidson and MacKinnon,
  - MLE for SEMs, see Lecture 8
  - Ch 12.2 SUR systems
  - Ch 15.3 Encompassing

VAR systems (unrestricted reduced forms) I

- With the notation from the FIML part of Lecture 8, consider the **open-VAR** (also known as **VARX**) that we typically specify when we do (explicit) Multiple-Equation Modelling:

$$\mathbf{y}_t = \sum_{i=1}^{p} \mathbf{\Pi}_i \mathbf{y}_{t-i} + \sum_{i=0}^{q} \mathbf{\Gamma}_i \mathbf{z}_{t-i} + \boldsymbol{\varepsilon}_t \quad (1)$$

$$\boldsymbol{\varepsilon}_t = IN(\mathbf{0}, \mathbf{\Sigma}) \quad (2)$$

$$t = 1, 2, \ldots, T \quad (3)$$

- In the program, this is called the " **Unrestricted System"** also called **"Unrestricted Reduced Form"**, URF
- Why should we want to "go beyond" the VAR?

VAR systems (unrestricted reduced forms) II

- May be interested in other “equation parameters” than $\mathbf{\Pi}_i$, and $\mathbf{\Gamma}_i$:
- The parameters of the identified structural equations and of conditional equations are parameters that are derived from the VAR parameters
- The **impulse responses (dynamic multipliers)** of random shocks.

The lack of identification of VAR impulse responses I

- Consider the bivariate case VARX(1)

$$\underbrace{\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}}_{\mathbf{\Pi}_1} \underbrace{\begin{pmatrix} Y_{1t-1} \\ Y_{2t-1} \end{pmatrix}}_{\mathbf{y}_{t-1}} + \underbrace{\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}}_{\mathbf{\Gamma}_0} \underbrace{\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}}_{\mathbf{z}_t} + \underbrace{\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}}_{\boldsymbol{\varepsilon}_t} \quad (4)$$

- After estimation of this VAR we can study the estimated partial derivatives of Y_{1t+j} and Y_{2t+j} with respect to shocks in ε_{1t} and ε_{2t}.

The lack of identification of VAR impulse responses II

- These estimated parameters are called **impulse-responses** (mathematically they are exactly like the *lag-weights* or *dynamic multipliers* from the final equation that can be solved out from the VAR (by setting $\mathbf{\Gamma}_0 = \mathbf{0}$).
- But we cannot identify, or give economic interpretation to, these impulse-responsens as for example the partial effects of supply or demand shocks.
- The reason is easily illustrated: Assume that we have a *SEM* that we write compactly as:

$$\mathbf{B}\mathbf{y}_t = \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Upsilon}_0 \mathbf{z}_t + \boldsymbol{\epsilon}_t \tag{5}$$

where the **structural disturbances** are $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \epsilon_{2t})'$, for example a **demand shock** ϵ_{1t} and a **supply-shock** ϵ_{2t}, if the

The lack of identification of VAR impulse responses III

first structural equation is a demand function, and the second structural equation is a supply function.

- The covariance between the structural disturbances is ω_{12}, the variances are ω_1^2 and ω_2^2.
- The VAR disturbances are then (in general) just a linear combination of the structural disturbances:

$$\boldsymbol{\varepsilon}_t = B^{-1}\boldsymbol{\epsilon}_t$$

i.e. they are reduced form disturbances and the estimated impulse responses from the VAR cannot be interpreted as the dynamics effect of unique demand and supply shocks.

- The VAR disturbances are (in general) correlated even if the structural disturbances are uncorrelated.

The lack of identification of VAR impulse responses IV

- To obtains estimates of identified impulse responses we need to go beyond the VAR. The models of the VAR that we concentrate on here is:
  - An **identified SEM**
  - A model made up of a system of **recursive equations.**

Since we have already discussed SEMs a good deal, we spend a little time on the recursive model (sometimes called *causal chain* model)

SVAR and recursive model I

- For simplicity set $\boldsymbol{\Upsilon}_0 = \mathbf{0}$ to give the SEM(1) (no loss of generality)

$$\begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix} \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Y_{1t-1} \\ Y_{2t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} \tag{6}$$

- The unrestricted reduced form, URF, i.e. the VAR is:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \frac{\phi_{21} b_{12}}{b_{12} b_{21} - 1} - \frac{\phi_{11}}{b_{12} b_{21} - 1} & \frac{\phi_{22} b_{12}}{b_{12} b_{21} - 1} - \frac{\phi_{12}}{b_{12} b_{21} - 1} \\ \frac{\phi_{11} b_{21}}{b_{12} b_{21} - 1} - \frac{\phi_{21}}{b_{12} b_{21} - 1} & \frac{\phi_{12} b_{21}}{b_{12} b_{21} - 1} - \frac{\phi_{22}}{b_{12} b_{21} - 1} \end{pmatrix} \begin{pmatrix} Y_{1t-1} \\ Y_{2t.1} \end{pmatrix}$$
$$+ \begin{pmatrix} -\frac{1}{b_{12} b_{21} - 1} \epsilon_{1t} + \frac{b_{12}}{b_{12} b_{21} - 1} \epsilon_{2t} \\ \frac{b_{21}}{b_{12} b_{21} - 1} \epsilon_{1t} - \frac{1}{b_{12} b_{21} - 1} \epsilon_{2t} \end{pmatrix}$$

SVAR and recursive model II

- If we impose the recursive structure: $b_{21} = 0 = \phi_{21} = 0$ we obtain the **structural VAR**, SVAR:

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} -\frac{\phi_{11}}{-1} & \phi_{22}\frac{b_{12}}{-1} - \frac{\phi_{12}}{-1} \\ 0 & -\frac{\phi_{22}}{-1} \end{pmatrix} \begin{pmatrix} Y_{1t-1} \\ Y_{2t-1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{-1}\epsilon_{1t} + \frac{b_{12}}{-1}\epsilon_{2t} \\ 0\epsilon_{1t} - \frac{1}{-1}\epsilon_{2t} \end{pmatrix}$$

$$\begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} - \phi_{22}b_{12} \\ 0 & \phi_{22} \end{pmatrix} \begin{pmatrix} Y_{1t-1} \\ Y_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon^*_{1t} \\ \varepsilon^*_{2t} \end{pmatrix} \tag{7}$$

- What is the difference between this SVAR and the VAR (URF)?
  - First: We have one-way Granger causality

SVAR and recursive model III

- Second: The SVAR disturbances ε^*_{1t} and ε^*_{2t} :

$$
\begin{aligned}
\varepsilon^*_{1t} &= \epsilon_{1t} - b_{12}\epsilon_{2t} \\
\varepsilon^*_{2t} &= \epsilon_{2t}
\end{aligned}
$$

  are uncorrelated if $b_{12} \equiv \omega_{12}/\omega_2^2$.

- This means that if $b_{12} \equiv \omega_{12}/\omega_2^2$, the impulse-responses with respect to ε^*_{1t} and ε^*_{1t} are identified.
- Now, go back to the SEM in (6) and see what $b_{21} = 0 = \phi_{21} = 0$ entail there:

$$
\begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} \end{pmatrix} \begin{pmatrix} Y_{1t-1} \\ Y_{2t-1} \end{pmatrix} + \begin{pmatrix} \epsilon^*_{1t} \\ \epsilon^*_{2t} \end{pmatrix} \tag{8}
$$

SVAR and recursive model IV

- We have **one-way causal ordering** both contemporaneously and in terms of Granger-Causality: There are two lower-triangular matrices with coefficients
- In addition we see that

$$\epsilon^*_{2t} \equiv \varepsilon^*_{1t} = \epsilon_{2t}$$

  since the second line in the recursive model (8) is identical to the second row of the structural VAR
- Moreover, if $b_{12} \equiv \omega_{12}/\omega_2^2$, the equation in the first row in the recursive model is a regression model of Y_{1t} given Y_{2t} and therefore

$$\epsilon^*_{1t} \equiv \varepsilon^*_{1t} = \epsilon_{1t} - \omega_{12}/\omega_2^2 \epsilon_{2t},$$

  so that $Cov(\epsilon^*_{1t}, \epsilon^*_{2t}) = 0$ in the recursive model.

SVAR and recursive model V

- **Conclusion**: The first equation in the recursive model (8) as the conditional model of Y_{1t} given Y_{2t}, and the second row as the marginal model for Y_{2t}, the disturbances of the recursive model is the same as the SVAR disturbances.
- **Implication for estimation**: OLS estimation, equation by equation, give efficient ML estimation of both the dynamic recursive model (8) and of the SVAR (7).
- The identification of the VAR disturbances by means of a recursive one-way causality is called (more technically) a Cholesky factorization in the VAR literature. But it amounts to the same thing.
- We also see that both the SVAR (7) and the recursive model (8) impose conditioning in a recursive order over equations.

SVAR and recursive model VI

- In most textbooks, the conditioning is kept implicit. The orthogonality of the disturbances is instead presented as a condition that must be satisfied for recursive models, in addition to the condition of triangular shapes of the coefficient matrices.
- This way if defining recursive models is more practical for systems with many varaibles and lags.

SVAR and recursive model VII

Class questions:

- Why is the recursive model (8) exactly identified, even though it fails on the order and rank conditions:
- Use orthogonality of disturbances as condition for a recursive model. How would you specify the recursive model if we start from:

$$\mathbf{B}\mathbf{y}_t = \mathbf{\Phi}_1\mathbf{y}_{t-1} + \mathbf{\Upsilon}_0\mathbf{z}_t + \boldsymbol{\epsilon}_t$$

where $\mathbf{y}_t$ is 3×1 and $\mathbf{z}_t$ 2×1?

Encompassing the VAR I

- Like SEMs, recursive dynamic models can also be overidentified.
- The **maximized likelihood** of (1) for the sample $t = 1, 2, \ldots, T$ denoted L_{URF} can be regard as a benchmark for such econometric models of the VAR.
- Therefore: For both SEMs and recursive dynamic models we can compare the **unrestricted reduced form** log likelihood value L_{URF} of the VAR with the **restricted** maximized log likelihood value L_{RRF} from a SEM or a recursive model
- The *LR test of over-identifying restrictions*:

$$-2(L_{RRF} - L_{URF}) \sim \chi^2(\text{number of overid restrictions})$$

can therefore be given a wider interpretation:

Encompassing the VAR II

- If the test is insignificant if gives proof that our model of the VAR (be it SEM or recursive) is explaining as much of the joint variability of the $\mathbf{y}_t$ vector as the VAR does, but with fewer estimated parameters.
- We say the our dynamic econometric model parsimoniously **encompasses the VAR**,

Encompassing more generally I

- Encompassing means "putting a fence around". In the case we have discussed above, economic interpretable econometric model puts a fence around the statistical model (the VAR):
- It predicts the $\mathbf{y}_t$ vector just as well as the VAR but with fewer estimated parameters.
- More generally, encompassing represents the methodological position that new models of a variable (or vector of variables) should explain the properties of preexisting models.
- More concrete:
  - Start a research project by reviewing existing models;
  - replicate the results of one or more of the models (obtain the data, estimate on the same data and with the same method)

Encompassing more generally II

- Present you own model, and make sure that you encompass previous studies

- There are many versions of encompassing tests. One of the most common is our friend the F-test of exclusion restrictions. This can be used even though the "old model" M1 and the "new model M2" are non-nested. How?
- Ch 15.3 in HG presents more of the theory of formal encompassing tests. under the headline of "Nonnested Linear Regression models".

Systems of regression equations I

- Refer back to Lecture 1 and the notation for a single regression model:
- Let $\mathbf{X}$ be a $n \times k$ matrix with the regressors of the model

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon \tag{9}$$

  where $\mathbf{y}$ and $\boldsymbol{\varepsilon}$ are $T \times 1$, $\boldsymbol{\varepsilon}$ is white-noise $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \sigma^2\mathbf{I}$, and the parameter vector $\boldsymbol{\beta}$ is $k \times 1$.
- Assume that all variables in $\mathbf{X}$ are strictly exogenous or predetermined, so that (9) is indeed a regression model.

Systems of regression equations II

- Assume that we have a system of g such regression models:

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i \quad i = 1, 2, \ldots, g \tag{10}$$
$$E(\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i') = \sigma_{ii}^2 \mathbf{I}$$

  where $\boldsymbol{\beta}_i$ is $k_i \times 1$.
- Examples: A system of demand functions; a VAR

SUR systems I

- The equations of the system of regression equations. are unrelated if there is no correlation between the disturbances of different equations
- However, the equations are only **Seemingly Unrelated Regressions** (SUR) if there are any non-zero correlations between the disturbances.
- As shown in HG Ch 12.2 the SUR system can be written compactly by stacking vectors and matrices, and it can be analysed by extending the usual matrix multiplication to the **Kronecker product** (details are given).
- In that notation, the SUR system has the same structure as a single regression model with $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \boldsymbol{\Sigma}$.

SUR systems II

- It then follows that there is a *system version* of the GLS estimator which is more efficient than the OLS estimator.
- This is called the SUR estimator.
- There are two cases where SURE becomes numerically identical to OLS (regression by regression)
  - There is no correlation between disturbances
  - All regressions have the same regressors: $\mathbf{X}_i = \mathbf{X}$ for all i.
- A proof for the second case in a lecture note later this week (for reference).
- An unrestricted VAR is a case of $\mathbf{X}_i = \mathbf{X}$ for all i.
- What about a recursive model? is that a SUR system?