RNy, econ4160 autumn 2013

Lecture note 6

GIVE and 2SLS equivalence.

The equivalence of $\hat{\boldsymbol{\beta}}_{GIV}$ and $\widehat{\boldsymbol{\beta}}_{2SLS}$ can be shown "both ways". DM page 323-324 has a short argument showing the equivalence by starting from $\widehat{\boldsymbol{\beta}}_{2SLS}$.

In the slide set to Lecture 6 we take the opposite direction. This brief note fills in a couple of steps. We start by GIVE:

$$\hat{\boldsymbol{\beta}}_{1,GIV} = (\widehat{\mathbf{W}}_1'\mathbf{X}_1)^{-1}\widehat{\mathbf{W}}_1'\mathbf{y}_1. \tag{1}$$

with

$$\widehat{\mathbf{W}}_1 = \begin{pmatrix} \mathbf{Z}_1 & : & \widehat{\mathbf{Y}}_1 \end{pmatrix}, \tag{2}$$

$$\mathbf{X}_1 = \begin{pmatrix} \mathbf{Z}_1 & : & \mathbf{Y}_1 \end{pmatrix} \tag{3}$$

and

$$\widehat{\mathbf{Y}}_1 = \mathbf{P}_{W_1}\mathbf{Y}_1. \tag{4}$$

where the matrices are given in the slide set, in particular:

$$\mathbf{P}_{W_1} = \mathbf{W}_1(\mathbf{W}_1'\mathbf{W}_1)^{-1}\mathbf{W}_1' \tag{5}$$

Using the partitioning, the product $\widehat{\mathbf{W}}_1'\mathbf{X}_1$ becomes:

$$\begin{aligned}\widehat{\mathbf{W}}_1'\mathbf{X}_1 &= \begin{pmatrix} \mathbf{Z}_1 & : & \widehat{\mathbf{Y}}_1 \end{pmatrix}' \times \begin{pmatrix} \mathbf{Z}_1 & : & \mathbf{Y}_1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{Z}_1' \\ \widehat{\mathbf{Y}}_1' \end{pmatrix} \begin{pmatrix} \mathbf{Z}_1 & : & \mathbf{Y}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_1'\mathbf{Z}_1 & \mathbf{Z}_1'\mathbf{Y}_1 \\ \widehat{\mathbf{Y}}_1'\mathbf{Z}_1 & \widehat{\mathbf{Y}}_1'\mathbf{Y}_1 \end{pmatrix}\end{aligned}$$

so

$$\hat{\boldsymbol{\beta}}_{1,GIV} = \begin{pmatrix} \mathbf{Z}_1'\mathbf{Z}_1 & \mathbf{Z}_1'\mathbf{Y}_1 \\ \widehat{\mathbf{Y}}_1'\mathbf{Z}_1 & \widehat{\mathbf{Y}}_1'\mathbf{Y}_1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{Z}_1'\mathbf{y}_1 \\ \widehat{\mathbf{Y}}_1'\mathbf{y}_1 \end{pmatrix}$$

We now focus on $\mathbf{Z}_1'\mathbf{Y}_1$ and $\widehat{\mathbf{Y}}_1'\mathbf{Y}_1$. Start with the second:

$$\widehat{\mathbf{Y}}_1'\mathbf{Y}_1 = (\mathbf{P}_{\mathbf{W}_1}\mathbf{Y}_1)'\mathbf{Y}_1$$

The reduced form residual $\mathbf{e}_1$ matrix is given by

$$\mathbf{e}_1 \equiv \mathbf{Y}_1 - \underbrace{\mathbf{P}_{W_1}\mathbf{Y}_1}_{\widehat{\mathbf{Y}}_1} = \mathbf{M}_{W_1}\mathbf{Y}_1$$

where

$$\mathbf{M}_{W_1} = \mathbf{I} - \mathbf{P}_{W_1}$$

Using this we get

$$\begin{aligned}(\mathbf{P}_{\mathbf{W}_1}\mathbf{Y}_1)'\mathbf{Y}_1 &= (\mathbf{P}_{\mathbf{W}_1}\mathbf{Y}_1)'(\widehat{\mathbf{Y}}_1 + \mathbf{e}_1) \\ &= \mathbf{Y}_1'\mathbf{P}_{\mathbf{W}_1}'\widehat{\mathbf{Y}}_1 = \widehat{\mathbf{Y}}_1'\widehat{\mathbf{Y}}_1\end{aligned} \tag{6}$$

since $(\mathbf{P}_{\mathbf{W}_1}\mathbf{Y}_1)'\mathbf{e}_1 = \mathbf{Y}_1'\mathbf{P}_{\mathbf{W}_1}'(\mathbf{I}-\mathbf{P}_{\mathbf{W}_1})\mathbf{Y}_1 = \mathbf{0}$.[1]

Next, consider $\mathbf{Z}_1'\mathbf{Y}_1$. We want to show that $\widehat{\mathbf{Y}}_1'\mathbf{Z}_1 = \mathbf{Z}_1'\mathbf{Y}_1$. We can start by writing $\widehat{\mathbf{Y}}_1'\mathbf{Z}_1$:

$$\widehat{\mathbf{Y}}_1'\mathbf{Z}_1 = (\mathbf{Y}_1' - \mathbf{e}_1')\mathbf{Z}_1 = \mathbf{Y}_1'\mathbf{Z}_1 - \mathbf{e}_1'\mathbf{Z}_1$$

Here $\mathbf{e}_1'\mathbf{Z}_1 = \mathbf{0}$ (since the effect of the exogenous $\mathbf{Z}_1$ has been regressed out, but try to show formally if you want). But then

$$\widehat{\mathbf{Y}}_1'\mathbf{Z}_1 = \mathbf{Y}_1'\mathbf{Z}_1 \Leftrightarrow \mathbf{Z}_1'\widehat{\mathbf{Y}}_1 = \mathbf{Z}_1'\mathbf{Y}_1 \tag{7}$$

Replacing $\widehat{\mathbf{Y}}_1'\mathbf{Y}_1$ by $\widehat{\mathbf{Y}}_1'\widehat{\mathbf{Y}}_1$ and $\mathbf{Z}_1'\mathbf{Y}_1$ by $\mathbf{Z}_1'\widehat{\mathbf{Y}}_1$ we get

$$\hat{\boldsymbol{\beta}}_{1,GIV} = \begin{pmatrix} \mathbf{Z}_1'\mathbf{Z}_1 & \mathbf{Z}_1'\widehat{\mathbf{Y}} \\ \widehat{\mathbf{Y}}_1'\mathbf{Z}_1 & \widehat{\mathbf{Y}}_1'\widehat{\mathbf{Y}}_1 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\mathbf{Y}}_1'\mathbf{y}_1 \\ \mathbf{Z}_1'\mathbf{y}_1 \end{pmatrix}$$

which is the expression for $\hat{\boldsymbol{\beta}}_{1,2SLS}$.

[1] Remember:

$$\mathbf{P}_{\mathbf{W}_1}' = \mathbf{P}_{\mathbf{W}_1}$$
$$\mathbf{P}_{\mathbf{W}_1}'\mathbf{P}_{\mathbf{W}_1} = \left[\mathbf{W}_1(\mathbf{W}_1'\mathbf{W}_1)^{-1}\mathbf{W}_1'\right]'\left[\mathbf{W}_1(\mathbf{W}_1'\mathbf{W}_1)^{-1}\mathbf{W}_1'\right] = \mathbf{P}_{\mathbf{W}_1}$$