Answer to "Lecture question" in Lecture 5

In Lecture 5 under the heading **Final equation**, we studied the Gaussian VAR

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix}, \tag{1}$$

where $\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$ is the matrix of autoregressive coefficients and we assume that

$$\begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix} \sim IID \left(\mathbf{0}, \begin{matrix} \sigma_y^2 & \sigma_{yx} \\ \sigma_{yx} & \sigma_x^2 \end{matrix} \right) \forall\ t \tag{2}$$

As an exercise, you were asked to show that (1) can be reduced to the so called **final equation** for Y_{t+1}

$$Y_{t+1} = (\underbrace{\pi_{11} + \pi_{22}}_{\equiv \rho_1})Y_t + (\underbrace{\pi_{12}\pi_{21} - \pi_{22}\pi_{11}}_{\equiv \rho_2})Y_t + \underbrace{\varepsilon_{yt+1} - \pi_{22}\varepsilon_{yt} + \pi_{12}\varepsilon_{xt}}_{\equiv \varepsilon_t}. \tag{3}$$

Try by repeated substitution: The answer is on tne next page!

Answer:

Solve the equation in the first row in (1) for X_{t-1}:

$$X_{t-1} = (1/\pi_{12})Y_t - (\pi_{11}/\pi_{12})Y_{t-1} - (1/\pi_{12})\varepsilon_{yt}. \quad (4)$$

Substitution in the second row of (1) gives

$$X_t = \frac{\pi_{22}}{\pi_{12}}Y_t + (\pi_{21} - \pi_{22}\frac{\pi_{11}}{\pi_{12}})Y_{t-1} - \frac{\pi_{22}}{\pi_{12}}\varepsilon_{yt} + \varepsilon_{xt},\ \pi_{12} \neq 0. \quad (5)$$

Finally: Change t to $t+1$ in the first row of (1), and replace X_t by the expression on the right hand side of (5):

$$Y_{t+1} = \pi_{11}Y_t + \pi_{12}\left(\frac{\pi_{22}}{\pi_{12}}Y_t + (\pi_{21} - \pi_{22}\frac{\pi_{11}}{\pi_{12}})Y_{t-1} - \frac{\pi_{22}}{\pi_{12}}\varepsilon_{yt} + \varepsilon_{xt}\right) + \varepsilon_{yt+1} \quad (6)$$

Collecting terms gives

$$Y_{t+1} = (\pi_{11} + \pi_{22})Y_t + (\pi_{12}\pi_{21} - \pi_{22}\pi_{11})Y_{t-1} + \varepsilon_{yt+1} - \pi_{22}\varepsilon_{yt} + \pi_{12}\varepsilon_{xt}, \quad (7)$$

which is what we should find. Clearly this equation must hold for all periods, so we can write

$$Y_t = (\pi_{11} + \pi_{22})Y_{t-1} + (\pi_{12}\pi_{21} - \pi_{22}\pi_{11})Y_{t-2} + \varepsilon_{yt} - \pi_{22}\varepsilon_{yt-1} + \pi_{12}\varepsilon_{xt-1}, \quad (8)$$

the **final equation** for Y_t defined by the system (1).

A final equation expresses an endogenous variable by the "its own lags" and exogenous random variables. No lags of other endogenous variables are allowed in a final equation expression.

Additional questions, for review:

1. Write down the homogenous difference equation that correponds to the final equation for Y_t.

2. Write down the associated characteristic equation. How does the characteristic roots of the homogenous equation relate to the conditions for dynamic stability and covariance stationarity of Y_t?

3. Derive the final equation for X_t. What are the requirements for stationarity of X_t.

4. What are the conditions for stationariy of the vector time-series $\mathbf{y}_t = (Y_t, X_t)'$?