

Lecture note 4

Single equation model typology.

This is my notes that supplement the end of Lecture 5, by introducing a single equation model typology that we also will review during the computer classes.

ADL(1,1)

We loose nothing by considering the case with only a single conditioning explanatory variable (therefore we can drop the subscripts for variable number):

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \varepsilon_t \quad (1)$$

where is ε_t ($t = 1, 2, \dots, T$) are a sequence of white-noise disturbances. See HN § 14.3 for a full statistical specification of the ADL model.

Several models that are used in applied econometrics are either special cases (simplifications) of (1), and one is a re-parameterization. This note only gives the main models in such a typology.

Static model

If the joint hypothesis $\phi_1 = \beta_1 = 0$ is true, (1) simplifies to

$$Y_t = \phi_0 + \beta_0 X_t + \varepsilon_t \quad (2)$$

without affecting the statistical properties of ε_t . If $\phi_1 = \beta_1 = 0$ does not hold, the disturbance of static regression between Y_t and X_t can logically not be independent of the information set Y_{t-1} , X_t and X_{t-1} , even if it is uncorrelated with X_t alone (cf first lecture).

Model in differences

If the joint hypothesis $\phi_1 = 1$ and $\beta_1 = -\beta_0$ are true, (1) simplifies to

$$\Delta Y_t = \phi_0 + \beta_0 \Delta X_t + \varepsilon_t \quad (3)$$

where the difference operator Δ is defined as $\Delta = 1 - L$. Note that something that resembles a “non-stationary condition” is now imposed in the form of $\phi_1 = 1$. Hence, if (3) is solved for Y_{t+1} , Y_{t+2} , \dots , with Y_t as an initial condition and given forward values of X_{t+j} and ε_{t+j} (typically 0), Y_{t+j} ($j = 1, 2, \dots$) will never reach a steady-state. Nevertheless, given that $\beta_1 = -\beta_0$ also holds in the DGP it is perfectly legitimate to estimate (3) since the properties of ε_t are the same as in the ADL.

Common factor model

(Davidson and MacKinnon has a quite extensive discussion of this model, starting on page 294.)

We start by expressing (1) by the lag-operator:

$$(1 - \phi_1 L)Y_t = \phi_0 + (\beta_0 + \beta_1 L)X_t + \varepsilon_t.$$

and factorizing the two lag-polynomials

$$\begin{aligned} (1 - \phi_1 L) &= \phi^*(L)\phi^{**}(L) \\ (\beta_0 + \beta_1 L) &= \beta^*(L)\beta^{**}(L) \end{aligned}$$

where $\phi^*(L) = 1$, $\phi^{**}(L) = 1 - \phi_1 L$, $\beta^*(L) = \beta_0$ and $\beta^{**}(L) = 1 + (\beta_1/\beta_2)L$. If the restriction $\phi^{**}(L) = \beta^{**}(L)$ is true, the two polynomials have a common factor. Written out, it is:

$$(1 - \phi_1 L) = (1 + \frac{\beta_1}{\beta_0} L)$$

or

$$(\beta_0 + \beta_1 L) = (1 - \phi_1 L)\beta_0 \quad (4)$$

In this case a simplification of (1) becomes:

$$Y_t = \frac{1}{1 - \phi_1 L} \phi_0 + \beta_0 X_t + \frac{1}{1 - \phi_1 L} \varepsilon_t$$

or

$$Y_t = \eta + \beta_0 X_t + u_t, \quad u_t = \phi_1 u_{t-1} + \varepsilon_t \quad (5)$$

where $\eta = \phi_0/(1 - \phi_1)$ and u_t is a the disturbance $u_t \sim AR(1)$ as you can see. In PcGive, (4) can be tested in the Test-Dynamic Analysis Menu.

Equilibrium correction model (ECM)

HN § 14.2

While the three first models are special cases or simplifications of (1), the ECM is a re-parameterization that always hold, it does not affect the time series properties of Y_t ($t = 1, 2, \dots$) or of ε_t ($t = 1, 2, \dots$). There are no restrictions imposed on the ADL(1,1).

If we subtract Y_{t-1} on each side of (1), and add and subtract $\beta_0 X_{t-1}$ on the right hand side we obtain

$$\Delta Y_t = \phi_0 + \beta_0 \Delta X_t + (\phi_1 - 1)Y_{t-1} + (\beta_0 + \beta_1)X_{t-1} + \varepsilon_t \quad (6)$$

which is called the equilibrium correction form of the ADL, or the error correction form of the ADL.

Equilibrium correction form is most precise since we can rewrite the part of the equation that holds the lagged levels to obtain:

$$\Delta Y_t = \beta_0 \Delta X_t + (\phi_1 - 1) \left\{ Y_{t-1} - \underbrace{\frac{\phi_0}{(1 - \phi_1)} - \frac{(\beta_0 + \beta_1)}{(1 - \phi_1)} X_{t-1}}_{Y_{t-1}^*} \right\} + \varepsilon_t \quad (7)$$

where the term inside the bracket is interpreted as deviation from equilibrium Y_t^* .

AR(1)

$\beta_0 = \beta_1 = 0$ gives the special case of an AR(1) for Y_t . Note that AR(1) is also an ECM, but the correction is now with respect to the unconditional expectation. The solution from Lecture 3 can be written as:

$$Y_t = \underbrace{\frac{\phi_0}{1 - \phi_1}}_{Y^* = E(Y_t)} + \left\{ Y_0 - \underbrace{\frac{\phi_0}{1 - \phi_1}}_{Y^* = E(Y_t)} \right\} \phi_1^t + \sum_{i=0}^{t-1} \phi_1^i \varepsilon_{t-i}. \quad (8)$$

to make the ECM interpretation even clearer. Later in the term this will help us understand the properties of model based forecasts, so we will come back to it then.

ADL(p,q) and ECM

All of the above can be extended and generalized. We can for example look at the ECM version of ADL(4,4) :

$$Y_t - \sum_{i=1}^4 \phi_i Y_{t-i} = \phi_0 + \sum_{i=0}^4 \beta_i X_{t-i} + \varepsilon_t, \quad (9)$$

one possibility is to put the levels term at the fourth lag

$$\begin{aligned} \Delta Y_t &= \phi_0 + \sum_{i=1}^3 \phi_i^\dagger \Delta Y_{t-i} + \sum_{i=0}^3 \beta_i^\dagger \Delta X_{t-i} \\ &+ (\phi(1) - 1)Y_{t-4} + \beta(1)X_{t-4} + \varepsilon_t \end{aligned} \quad (10)$$

where $\phi(1)$ is $\phi(L)$ with $L = 1$, $\beta(1)$ is the same for $\beta(L)$. Check that the coefficients of the lag-polynomials of ΔY_{t-j} and ΔX_{t-j} are:

$$\begin{aligned} \phi_i^\dagger &= \sum_{j=1}^i \phi_j - 1, \quad i = 1, 2, 3, \\ \beta_i^\dagger &= \sum_{j=0}^i \beta_j, \quad i = 1, 2, 3. \end{aligned} \quad (11)$$

Alternatively, we can place the level-terms at the first-lag

$$\begin{aligned} \Delta Y_t &= \phi_0 + \sum_{i=1}^3 \phi_i^\ddagger \Delta Y_{t-i} + \sum_{i=0}^3 \beta_i^\ddagger \Delta X_{t-i} \\ &+ (\phi(1) - 1)Y_{t-1} + \beta(1)X_{t-1} + \varepsilon_t \end{aligned} \quad (12)$$

$$\begin{aligned} \phi_i^\ddagger &= - \sum_{j=i+1}^4 \phi_j, \quad i = 1, 2, 3, \\ \beta_0^\ddagger &= \beta_0 \\ \beta_i^\ddagger &= - \sum_{j=i+1}^4 \beta_j, \quad i = 1, 2, 3. \end{aligned} \quad (13)$$

Check!. In both cases the long-run multiplier with respect to X_t is

$$K_1 = \frac{\beta(1)}{1 - \phi_1(1)} = \frac{\sum_{j=0}^4 \beta_j}{(1 - \sum_{j=1}^4 \phi_j)} \quad (14)$$

$$\beta_0^\dagger = \beta_0^\ddagger = \beta_0, \quad (15)$$

but note that the other coefficients of the model are not the same in the two versions.

$$\phi_i^\dagger \neq \phi_i^\ddagger, \quad i = 1, 2, 3, \quad (16)$$

$$\beta_i^\dagger \neq \beta_i^\ddagger, \quad i = 1, 2, 3. \quad (17)$$

ECMs are more flexible than this.

- The AR lag length and the Distributed Lag lengths need not be the same for the different variables.
- The levels of Y and X can be on different lags.

The long run multiplier K_1 is however invariant to the different ways of writing the ECM, it is the coefficients of the lags of ΔX_t and ΔY_t that are affected (as illustrated). Extension of the above to more than one explanatory variables (the k regressor case) is straight forward.

QUESTION: How can you obtain an estimator \hat{K}_1 , and obtain $\widehat{Var}(\hat{K}_1)$, from OLS estimation of ADL (1,1)?