Exam in ECON3220/4220, Fall 2020

Problem 1 (*Microeconomics*)

November 25, 2020

Part A: consumer theory (weight 14%, equally shared)

1. For each of the axioms on preferences in Chapter 1 (completeness, transitivity, continuity, strict monotonicity, and strict convexity), represent graphically preferences that do not satisfy that axiom.

2. Find the indirect utility function associated to the utility function $U(x_1, x_2) = x_1^\alpha x_2^\beta$ with $\alpha, \beta > 0$ and check Roy's identity for good 2.

SOLUTIONS

A1.

Completeness. Draw no-worse than and no-better than sets that do not cover the entire domain of preferences (typically positive orthant in 2 dimensions, but one can make a 1 dimension representation).

Transitivity. For example when preferences over two commodities are such that $(x_1, x_2) \succsim (z_1, z_2)$ if and only if $x_1 + x_2 - 2 \geq z_1 + z_2$. Then, $(0,0) \succsim (1,1) \succsim (2,2)$ but $(2,2) \succ (0,0)$. Graphically, the no-worse than set for $(1,1)$ is not a subset of the no-worse than set of $(2,2)$.

Continuity. Draw an indifference curve with a dotted line (suggesting there is no indifference)

Strict monotonicity. Draw preferences with an increasing indifference curve.

Strict convexity. Linear (or non-convex) preferences.

A2.

The demand functions solve the standard utility maximization problem:

$$\max_{x_1, x_2} x_1^\alpha x_2^\beta$$

$$\text{s.t.} p_1 x_1 + p_2 x_2 \leq y$$

For simplicity, I take the logaritmic transformation of the utility. Lagrangean:

$$\mathcal{L} = \alpha \ln x_1 + \beta \ln x_2 + \lambda (y - p_1 x_1 - p_2 x_2)$$

The solution is interior because of monotonicity of preferences and infinite marginal utility when a good goes to zero. The first order conditions are:

$$\begin{cases} \frac{\alpha}{x_1} = \lambda p_1 \\ \frac{\beta}{x_2} = \lambda p_2 \\ p_1 x_1 + p_2 x_2 = y \end{cases}$$

Combining the first two gives:

$$x_2 = \frac{\beta p_1}{\alpha p_2} x_1$$

Substituting in the budget constraint gives the demand for good 1:

$$x_1(p, y) = x_1^* = \frac{\alpha}{\alpha + \beta} \frac{y}{p_1}$$

Substituting in the previous gives:

$$x_2(p, y) = x_2^* = \frac{\beta}{\alpha + \beta} \frac{y}{p_2}$$

Substituting in the original utility function gives:

$$V(p, y) = U(x_1(p, y), x_2(p, y)) = \left(\left(1 + \frac{\beta}{\alpha}\right)^{-1} \frac{y}{p_1}\right)^{\alpha} \left(\left(1 + \frac{\alpha}{\beta}\right)^{-1} \frac{y}{p_2}\right)^{\beta}$$

$$= \frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha+\beta}} \frac{y^{\alpha+\beta}}{p_1^\alpha p_2^\beta}$$

To check Roy's identity for good 2, we need the derivative of the indirect utility function with respect to p_2 and y.

$$\frac{\partial}{\partial p_2} V(p, y) = -\beta \frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha+\beta}} \frac{y^{\alpha+\beta}}{p_1^\alpha p_2^\beta} p_2^{-1}$$

$$\frac{\partial}{\partial y} V(p, y) = (\alpha + \beta) \frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha+\beta}} \frac{y^{\alpha+\beta}}{p_1^\alpha p_2^\beta} y^{-1}$$

Roy's identity tells that the demand for good two is given by:

$$x_2(p, y) = -\frac{\frac{\partial}{\partial p_2} V(p, y)}{\frac{\partial}{\partial y} V(p, y)} = -\frac{-\beta \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta}} \frac{y^{\alpha+\beta}}{p_1^\alpha p_2^\beta} p_2^{-1}}{(\alpha + \beta) \frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{\alpha+\beta}} \frac{y^{\alpha+\beta}}{p_1^\alpha p_2^\beta} y^{-1}}$$

$$= -\frac{-\beta p_2^{-1}}{(\alpha + \beta) y^{-1}} = \frac{\beta}{\alpha + \beta} \frac{y}{p_2}$$

Part B: production theory (weight 15%, equally shared)

1. Explain the economic meaning of concavity with respect to prices of the cost function.

2. For each of the following, show that these cannot be cost functions (these violate some of its properties):
   a) $c(\boldsymbol{w}, y) = y^2 (w_1 + e^{w_2})$;
   b) $c(\boldsymbol{w}, y) = \sqrt{(1-y) w_1 w_2}$;
   c) $c(\boldsymbol{w}, y) = y \left(\frac{w_1}{w_2}\right)$;

3. For the cost function $c(\boldsymbol{w}, y) = y^2 w_1$, derive the production function and the profit function.

SOLUTIONS

B1.

Assume a price of a good changes. If the firm does not change inputs, the cost of the firm changes linearly with the price (the cost is given by quantity of inputs times price of inputs). However, the firm might reoptimize and reduce the cost. This means that the cost function is concave in prices.

B2.

a) the function is convex with respect to the price of input 2.
b) This function is decreasing with respect to y
c) this function is decreasing in the price of input 2.

B3.

The production function associated to this cost function is:

$$f(x_1, x_2) = \sqrt{x_1}.$$

The profit function is obtained by finding the optimal supply y and substituting back.

$$\max_y py - y^2 w_1$$

FOC:

$$p = 2y^* w_1$$

Optimal supply:

$$y(p, \boldsymbol{w}) = y^* = \frac{p}{2w_1}$$

Profit function:

$$\Pi(p, \boldsymbol{w}) = py^* - (y^*)^2 w_1 = \frac{p^2}{4w_1}.$$

Part C: general equilibrium and taxation (weight 21%, equally shared)

Corn is used for both the production of food and for the production of ethanol, an alternative fuel. Denote the quantities of corn by x_C, food by x_F, and ethanol by x_E.

There are two representative firms, operating respectively in the food and ethanol sectors. The production functions in both the food industry and the ethanol sectors are constant returns to scale. Let z_C denote the input of corn, y_F denote the output of food, and y_E the output of ethanol. Then, $y_F \leq f_F(z_C) = Az_C$ with $A > 0$ and $y_E \leq f_E(z_C) = z_C$.

The representative individual has a utility function $U(x_F, x_E) = \ln x_F + x_E$ and an endowment of corn ω_C.

The market is perfectly competitive.

1. Verify that the conditions of the first welfare theorem are satisfied.
2. Compute the general equilibrium outcome. [*Hint: the easiest method is to directly identify the optimal quantities, without using prices, demand, supply, etc.*]
3. Briefly explain the inverse elasticity rule for optimal commodity taxation and discuss its implications for the taxation of food and ethanol. *[Clarification: no need to find the optimal taxes nor the elasticities!]*

SOLUTIONS

C1.

For the first welfare theorem to hold, we need that preferences are locally non satiated. The utility is strictly increasing in the commodities.

C2.

Since the competitive equilibrium is Pareto efficient, we can find the set of Pareto efficient allocations by maximizing the utility on the feasible allocations.

$$\max_{x_F, x_E} \ln x_F + x_E$$

subject to:

$$x_F \leq Az_{CF}$$

$$x_E \leq z_{CE}$$

$$z_{CF} + z_{CE} \leq \omega,$$

where z_{CF} is the input of corn for food production and z_{CE} is the input of corn for ethanol production. Substituting gives:

$$\max_{z_{CF}} \ln Az_{CF} + (1 - z_{CF}).$$

The first order condition for an interior solution is then:

$$\frac{1}{A} = z_{CF},$$

but this holds only if $A \geq 1$. If $A < 1$, a corner solution will emerge with $z_{CF} = 1$.

Thus, the unique equilibrium outcome is:

$$z_{CF} = \begin{cases} \frac{1}{A} & \text{if } A \geq 1 \\ 1 & \text{if } A < 1 \end{cases}$$

$$y_F = x_F = \begin{cases} 1 & \text{if } A \geq 1 \\ A & \text{if } A < 1 \end{cases}$$

$$y_E = x_E = \begin{cases} 1 - \frac{1}{A} & \text{if } A \geq 1 \\ 0 & \text{if } A < 1 \end{cases}$$

C3.

If the demand of a commodity is independent of the price of others (cross price elasticities are zero), the inverse elasticity rule tells that the tax rate on a consumption good is inversely proportional to its elasticity.

When $A < 1$, the only good purchased in food, so it is the only one that can be taxed. As A increases, the demand for food becomes more elastic and taxes start to be levied also on ethanol.

Solution sketches for the exam in ECON3220/4220, Fall 2020

Problem 2 (Game theory and the economics of information)

Part A: Rationalizable strategies and mixed-strategy Nash equilibrium

Weight: 16% *(with equal weight* = 8% *on each subproblem)*

Consider the following normal form game.

	L	C	R
U	0, 0	2, 0	1, 4
M	–1, 4	3, 1	3, 2
D	2, 0	2, 3	0, 1

(a) What strategies are rationalizable for each of the two players?

A strategy is rationalizable if and only if it survives iterated elimination of strictly dominated strategies. U is strictly dominated by a mixture of M and D where the probabilities of each of these strategies exceeds $\frac{1}{3}$. With U eliminated, R is strictly dominated by a mixture of L and C where the probabilities of each of these strategies exceeds $\frac{1}{3}$. No further elimination is possible. Hence, M and D are rationalizable for player 1, L and C are rationalizable for player 2.

(b) The game has one and only one Nash equilibrium, in mixed strategies. Find the mixed strategies that the players use in this Nash equilibrium.

By determining, for each player, the best response to pure strategies for the opponent, it can easily be checked that there is no Nash equilibrium in pure strategies. Since U is strictly dominated, this strategy cannot be a best response to any belief concerning the choice of player 2 and cannot be assigned positive probability in a mixed-strategy Nash equilibrium. If U is played with probability 0, then R cannot be a best response to any remaining belief concerning the choice of player 1 and cannot be assigned positive probability in a mixed strategy Nash equilibrium. For player 1 to be indifferent between M and D, player 2 must assign probability $\frac{1}{4}$ to L and probability $\frac{3}{4}$ to C. For player 1 to be indifferent between L and C, player 2 must assign probability $\frac{1}{2}$ to M and probability $\frac{1}{2}$ to D. Hence, the mixed strategy that player 1 uses is $(0, \frac{1}{2}, \frac{1}{2})$ and the mixed strategy that player 2 uses is $(\frac{1}{4}, \frac{3}{4}, 0)$.

Part B: Auctions under complete and incomplete information

Weight: 24% *(with equal weight* = 8% *on each subproblem)*

Consider a private value second-price sealed-bid auction with two bidders who both are risk-neutral, have valuations in the interval $[0, 1]$ and can submit bids in the interval $[0, 1]$. In a private value second-price sealed-bid auction the bidders are first informed of their valuations, they then submit their bids, the item is assigned to the bidder with the highest bid (and with equal probability to each if the two bids are equal), and the winner pays the second highest bid (i.e., the bid of the other bidder when there are only two bidders). The utility of the winner equals his/her valuation minus the payment, while the other bidder has zero utility.

(a) Assume first that the valuations of the two player are commonly known, with $v_1 = 1$ and $v_2 = 0$. One Nash equilibrium of the auction is that the players bid their valuations ($b_1 = 1$ and $b_2 = 0$), as bidder 1's utility equals 1 for any positive bid and 1 with probability equal to 0.5 for a bid equal to 0, and bidder 2's utility equals 0 for any bid less than 1 and -1 with probability equal to 0.5 for a bid equal to 1.

Another Nash equilibrium is that $b_1 = 0$ and $b_2 = 1$. Why?

Given that $b_2 = 1$, player 1 cannot gain by bidding 1 and thereby having equal chance of winning the auction as his/her valuation is 1 and the payment if the auction is won is 1. Given that $b_1 = 0$, the utility of player 2 is 0 if he/she wins the auction as his/her valuation is 0 and the payment is 0. Therefore, player 2 cannot gain by bidding 0 and thereby having equal chance of not winning the auction.

(b) Assume now that the valuations are independent and identically distributed on the interval $[0, 1]$, and each bidder is only informed of his/her own valuation. Consider that each player $i = 1, 2$ bids his/her valuation, i.e., $b_1(v_1) = v_1$ for all v_1 in $[0, 1]$ and $b_2(v_2) = v_2$ for all v_2 in $[0, 1]$. Why is this a Bayesian Nash equilibrium?

Consider the strategy of player i and let j be the other player. If j bids less that v_i, then player i gains by winning the auction, and bidding v_i achieves this. The utility when winning does not depend on his/her own bid as long as he/she wins, so v_i is an optimal choice. If j bids at least v_i, then player i cannot gain by winning the auction, and bidding v_i ensures that his/her utility is 0. Hence, $b_i(v_i) = v_i$ for all v_i in $[0, 1]$ is a best response to any bidding strategy for the other bidder j and thus also to the strategy $b_j(v_j) = v_j$ for all v_j in $[0, 1]$.

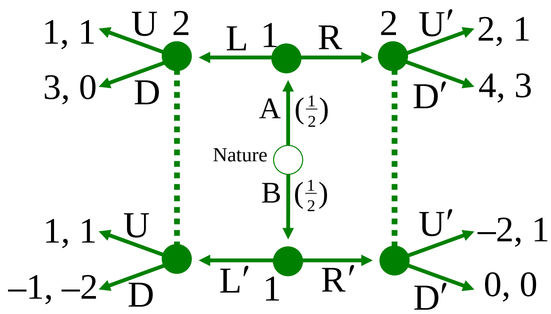
(c) Assume still, as in part (b), that the valuations are independent and identically distributed on the interval $[0, 1]$, and each bidder is only informed of his/her own valuation. But consider now that bidder 1 always bids 0, independently of his/her valuation, i.e., $b_1(v_1) = 0$ for all v_1 in $[0, 1]$, and that bidder 2 always bids 1, independently of his/her valuation, i.e., $b_2(v_2) = 1$ for all v_2 in $[0, 1]$. Why is this also a Bayesian Nash equilibrium?

Given that $b_2(v_2) = 1$ for all v_2 in $[0, 1]$, player 1, independently of his/her valuation, cannot gain by bidding 1 and thereby having equal chance of winning the auction as his/her valuation is at most 1 and the payment if the auction is won is 1. Given that $b_1(v_1) = 0$ for all v_1 in $[0, 1]$, the utility of player 2 is non-negative if he/she wins the auction as his/her valuation is non-negative and the payment is 0. Therefore, player 2, independently of his/her valuation, cannot gain by bidding 0 and thereby having equal chance of not winning the auction.

Part C: Perfect Bayesian equilibrium

Weight: 10%

Consider the following extensive form game, where the first number is the payoff of player 1 and the second number is the payoff of player 2. Show that this game has one and only one Perfect Bayesian equilibrium.



Player 2 will always choose U after having observed L/L′. Therefore, player 1 of type A will choose R, since the smallest payoff of doing so exceeds the payoff of 1 of choosing L. And player 1 of type B will choose L′, since the payoff of 1 of choosing L′ exceeds the largest payoff of choosing R′. It now follows from Bayes' rule that player 2 will assign probability 1 to player 1 being of type A after having observed R/R′ and therefore choose D′ in this case.