Exercise 4.4 and 4.5
4.4.1 Start with

$$C_1(F) = \int_0^1 u dM(u) = \int_0^1 u M'(u)\, du$$

By integration by parts, we know that:

$$uM(u)|_0^1 = \int_0^1 M(u)\, du + \int_0^1 uM'(u)\, du$$

and substituting for the Lorenz curve:

$$u\frac{L(u)}{u}\bigg|_0^1 = L(u)|_0^1 = 1 - 0 = \int_0^1 \frac{L(u)}{u} du + \int_0^1 uM'(u)\, du$$

thus:

$$\int_0^1 uM'(u)\, du = 1 - \int_0^1 \frac{L(u)}{u} du$$

and, substituting,

$$C_1(F) = 1 - \int_0^1 \frac{L(u)}{u} du$$

Again use integration by parts to obtain that:

$$(1 + \ln u) L(u)|_0^1 = 1 = \int_0^1 (1 + \ln u) L'(u)\, du + \int_0^1 \frac{L(u)}{u} du.$$

Thus,

$$1 - \int_0^1 \frac{L(u)}{u} du = \int_0^1 (1 + \ln u) L'(u)\, du,$$

and substituting gives:

$$C_1(F) = \int_0^1 (1 + \ln u) L'(u)\, du$$

Now remember that:

$$L(u) = \frac{1}{\mu} \int_0^u F^{-1}(t)\, dt$$

thus

$$L'(u) = \frac{1}{\mu} F^{-1}(u).$$

Substituting:

$$\int_0^1 (1 + \ln u) L'(u)\, du = \frac{1}{\mu} \int_0^1 F^{-1}(u) (1 + \ln u)\, du.$$

But $\frac{1}{\mu}\int_0^1 F^{-1}(u)(1+\ln u)\,du = \frac{1}{\mu}\int_0^1 F^{-1}(u)\,du + \frac{1}{\mu}\int_0^1 F^{-1}(u)\ln u du$ and $\frac{1}{\mu}\int_0^1 F^{-1}(u)\,du = 1$. Thus,

$$C_1(F) = \frac{1}{\mu}\int_0^1 F^{-1}(u)(1+\ln u)\,du.$$

Now use again integration by parts:

$$\int_0^1 F^{-1}(u)(1+\ln u)\,du = F^{-1}(u)(u\ln(u))\Big|_0^1 - \int_0^1 u\ln u dF^{-1}(u),$$

and solving:

$$\int_0^1 F^{-1}(u)(1+\ln u)\,du = 0 - \int_0^1 u\ln u dF^{-1}(u),$$

Thus:

$$C_1(F) = -\frac{1}{\mu}\int_0^1 u\ln u dF^{-1}(u).$$

Finally, let us substitute for $u = F(x)$ and, correspondingly, let us change the extremes of integration to get:

$$C_1(F) = -\frac{1}{\mu}\int_0^\infty F(x)\ln F(x)\,dx$$

4.4.2 Start with

$$C_2(F) = G = 1 - 2\int_0^1 L(u)\,du$$

By integration by parts, we know that:

$$uL(u)\big|_0^1 = \int_0^1 L(u)\,du + \int_0^1 uL'(u)\,du$$

and substituting for $L'(u) = \frac{1}{\mu}F^{-1}(u)$ and solving:

$$1 - \frac{1}{\mu}\int_0^1 uF^{-1}(u)\,du = \int_0^1 L(u)\,du,$$

thus:

$$C_2(F) = 1 - 2\left[1 - \frac{1}{\mu}\int_0^1 uF^{-1}(u)\,du\right] = \frac{2}{\mu}\int_0^1 uF^{-1}(u)\,du - 1.$$

By integration by parts:

$$\frac{1}{\mu}u^2F^{-1}(u)\Big|_0^1 = \frac{2}{\mu}\int_0^1 uF^{-1}(u)\,du + \frac{1}{\mu}\int_0^1 u^2 dF^{-1}(u).$$

$$\frac{1}{\mu}\left(1-u^2\right)F^{-1}(u)\Big|_0^1=-\frac{2}{\mu}\int_0^1 uF^{-1}(u)\,du+\frac{1}{\mu}\int_0^1\left(1-u^2\right)dF^{-1}(u).$$

Thus:

$$C_2(F)=\frac{1}{\mu}\int_0^1\left(1-u^2\right)dF^{-1}(u)-1.$$

Now substitute for $u=F(x)$ (including the extremes of integration) and remembering that $\mu=\int_0^\infty(1-F(x))\,dx$:

$$C_2(F)=\frac{1}{\mu}\left[\int_0^\infty\left(1-F^2(x)\right)dx-\int_0^\infty(1-F(x))\,dx\right].$$

And solving:

$$C_2(F)=\frac{1}{\mu}\int_0^\infty\left[F(x)-F(x)^2\right]dx.$$

$$C_2(F)=\frac{1}{\mu}\int_0^\infty F(x)(1-F(x))\,dx.$$

4.4.3 is similar as the above.

4.5.1
We showed that:

$$C_1(F)=\frac{1}{\mu}\int_0^1 F^{-1}(u)(1+\ln u)\,du.$$

Since $\int_0^1 F^{-1}(u)\,du=\mu$, this is equivalent to:

$$C_1(F)=1-\frac{\int_0^1(-\ln u)F^{-1}(u)\,du}{\mu}.$$

Notice that $W_1(F)\equiv\int_0^1(-\ln u)F^{-1}(u)\,du$ is a social welfare function with $P'(u)=(-\ln u)$. Thus, the social welfare function can be decomposed as follows:

$$W_1(F)=\mu(1-C_1(F)).$$

4.5.2 and **4.5.3** are similar.