ECON 4310
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Labor Supply

1. *Static model, single individuals*

Let
U=deterministic part of an ordinal utility function,
C=disposable income=consumption
h= annual hours worked
L= leisure=8760-h
w= wage rate
k= non-wage income
T= tax function

The behavior of the individual follows from solving the following maximization problem

$$\text{Max}_h U(C, L)$$
subject to
$$(1)\ \ C \le wh + k - T(wh, k)$$
$$(2)\ \ L = 8760 - h$$
$$(3)\ \ h \ge 0$$

Let r(C,h)=r(C,8760-L) denote the shadow price of leisure which is defined as the marginal rate of substitution, that is

$$(4)\ \ r(C, h) = -\frac{\partial U / \partial h}{\partial U / \partial C}$$

Moreover, let m(h) denote the marginal wage rate when the individual is working h hours:

$$(5)\ \ m(h) = w(1 - \partial T / \partial(wh))$$

From (1) we observe that C is a function of h, to his end we have C(h).

We then get:

$$(6)\ \ \text{If } r(C(0), 0) \le m(0), \text{ then } h > 0,$$
$$\text{otherwise } h = 0$$

If h>0, then h is determined by

$$(7)\quad r(C(h),h) = m(h)$$

To obtain some specific results we assume that

$$(8)\quad U(C,h) = \alpha \frac{C^{\mu}-1}{\mu} + \beta \frac{(1-\nu h)^{\gamma}-1}{\gamma},$$

where

$$(9)\quad \nu = \frac{1}{8760}$$

Normalized annual hours, νh, varies between 0 and 1, and consequently normalized annual leisure, L=1-νh, varies from 1 (at maximum) to 0 at minimum.

If

$$(10)\ \{\mu,\gamma\} < 1,$$

then the utility function is quasi-concave (sufficient condition).

When

$$(11)\ \{\mu,\gamma\} \to 0$$

then the utility function converges to a log-linear utility function.

From this model we get the following labor supply elasticities (given h>0).

1) Uncompensated marginal wage elasticity (Cournot elasticity)

$$(12)\quad El[h:m] \equiv \frac{\partial h}{\partial m}\frac{m}{h} = \frac{1+(\mu-1)\frac{mh}{C}}{(1-\mu)\frac{mh}{C}+(1-\gamma)\frac{\nu h}{(1-\nu h)}}$$

Note that the denominator always is positive (provided that the utility function is quasi-concave, necessary condition). The numerator can be positive but also negative, which means that a backward bending supply curve is possible.

$$\text{For } \mu \geq 1 - \frac{C}{mh} = -\frac{I}{mh} < 0$$

labor supply is increasing (or non-decreasing) with the wage rate. For smaller μ (or given μ, for larger I/mh) labor supply is decreasing with the wage rate.

2) Compensated marginal wage elasticity (Slutsky elasticity)

$$(13)\quad El\,[h:m]\,|\,U\,\text{constant} = \frac{\partial h}{\partial m}\frac{m}{h}\,|\,(U=\bar{U}) = \frac{1}{(1-\mu)\frac{mh}{C} + (1-\gamma)\frac{\nu h}{(1-\nu h)}}$$

Provided that the utility function is quasi-concave, this elasticity is always positive.

3) Virtual income elasticity

$$(14)\quad El[h:I] = \frac{(\mu-1)\frac{I}{C}}{(1-\mu)\frac{mh}{C} + (1-\gamma)\frac{\nu h}{(1-\nu h)}}$$

where I is virtual income, which relates to the tax bracket implied by the non-wage income, k, and the optimal wage income, wh, and it is implicit defined by

(15) C=mh+I.

We observe that the virtual income elasticity is always negative (provided a quasi-concave utility function). We also observe that

$$(16)\ El\,[h:m] = El\,[h:m]\,|\,(U=\bar{U}) + \frac{mh}{I}El\,[h:I]$$

Thus, the uncompensated wage elasticity can be decomposed into two terms: the first term is the substitution effect (the Slutsky elasticity) and the second term is the income effect. The substitution effect is always positive and the income effect is always negative. As long as the substitution effect is larger than the numerical value of the income effect, labor supply is increasing with the marginal wage rate. When the income effect dominates over the substitution effect, labor supply is declining with the wage rate.

4) Marginal wage elasticity when the marginal utility of income is constant (Frisch elasticity)

From the separable utility function given in (8) we observe that the marginal utility of income is constant as long as C is constant. Hence we easily get

$$(17)\ \text{El}[h:m] \mid C\ \text{constant} = \frac{\partial h}{\partial m}\frac{m}{h} \mid (C=\bar{C}) = \frac{1}{1-\gamma}\frac{1-\nu h}{\nu h}$$

We observe that this elasticity is always positive (provided a quasi-concave utility function).

Numerical estimates based on Norwegian data from 1979, mean values:

Type of elasticity	**Males**	**Females**
Cournot	**0.51**	**1.57**
Slutsky	**0.52**	**1.59**
Virtual income	**-0.01**	**-0.02**
Frisch	**0.53**	**1.62**

Rolf Aaberge, John K. Dagsvik and Steinar Strøm: Labor supply responses and welfare effects of tax reforms, *Scandinavian Journal of Economics*, 1995, Vol 97, no 4, 635-659.

2. Two period labor supply model

We assume that the individual is maximizing discounted future utilities with respect to consumption and hours of work in the two periods, given a budget constraint. Or, formally

(18) $$\max_{\{C_1,C_2,h_1,h_2\}}[U(C_1,h_1)+\frac{1}{1+\rho}U(C_2,h_2)]$$

given

(19) $$C_1+\frac{C_2}{1+r}=w_1h_1+\frac{w_2h_2}{1+r}$$

where C_t and h_t are consumption and hours of work in period t, w_t is the real wage rate in period t, r is the rate of interest and ρ is the rate of time preference. The utility function U is, for example, the one specified in (8). Note that $1-\upsilon h_t=L_t$ where L_t is leisure. Maximum leisure equals 1.

The corresponding Lagrange function is given by

(20) $$£=U(C_1,h_1)+\frac{1}{1+\rho}U(C_2,h_2)-\lambda[C_1+\frac{C_2}{1+r}-w_1h_1-\frac{w_2h_2}{1+r}]$$

λ is the Lagrange coefficient. The optimal values for C_t and h_t are the solutions of the following first order conditions, together with eq (19):

(21) $$\frac{\partial U}{\partial C_1}-\lambda=0$$

(22) $$\frac{\partial U}{\partial C_2}\frac{1}{1+\rho}-\frac{\lambda}{1+r}=0$$

(23) $$\frac{\partial U}{\partial h_1}+\lambda w_1=0$$

(24) $$\frac{\partial U}{\partial h_2}\frac{1}{1+\rho}+\frac{\lambda w_2}{1+r}=0$$

From eqs (21)-(24) we get

(25) $$\frac{\partial U/\partial C_2}{\partial U/\partial C_1}=\frac{1+\rho}{1+r}$$

(26) $$-\frac{\partial U/\partial h_2}{\partial U/\partial h_1}=\frac{1+\rho}{1+r}\frac{w_2}{w_1}$$

From (25) and (26), and using the specification (8), we easily get

$$(27)\quad \left[\frac{1-\nu h_2}{1-\nu h_1}\right]^{(\gamma-1)} = \frac{1+\rho}{1+r}\frac{w_2}{w_1}$$

or, by using that $L_t = 1-\nu h_t$

$$(28)\quad \frac{L_1}{L_2} = \left[\frac{1+\rho}{1+r}\right]^{\frac{1}{1-\gamma}} \left[\frac{w_2}{w_1}\right]^{\frac{1}{1-\gamma}}$$

Thus

- if w_1 increases, hours of work in period 1, h_1, will increase relative to hours of work in period 2, h_2, (given $\gamma<1$),
- if both w_1 and w_2 rise by the same relative amount, then the ratio of leisure is unaffected,
- if r increases, hours of work in period 1, h_1, will increase relative to hours of work in period 2, h_2, (given $\gamma<1$).

3. Life-cycle model, single individuals

We will assume that the individuals have a perfect knowledge of all future variables like wage rates interest rates. Moreover we assume a constant interest rate and no rationing in credit markets. Subscript t (or τ) denotes period t (or τ). Taxation is ignored.

Let

U_t be the utility level, ρ rate of time preference, C_t consumption, N_t hours worked, L_t leisure, w_t the wage rate and A_0 initial wealth. T is the length of working life and retirement at T is assumed to be an exogenous event.

Optimal hours of work and consumption are assumed to be the solution of the following optimization problem.

$$(29)\quad \max_{C_t,h_t} \sum_{t=1}^{T}\left[\frac{1}{1+\rho}\right]^t U_t(C_t, L_t)$$

s.t.

$$(30)\quad A_0 + \sum_{t=1}^{T}\left[\frac{1}{1+r}\right]^t \left[w_t h_t - C_t\right] \geq 0$$

$$(31)\quad \nu h_t = 1 - L_t \geq 0$$

The first order conditions are

$$(32)\ \frac{\partial U_t}{\partial C_t} = \left[\frac{1+\rho}{1+r}\right]^t \lambda \ \ ; t = 1, 2, , , T$$

$$(33)\ \frac{\partial U_t}{\partial L_t} = \left[\frac{1+\rho}{1+r}\right]^t \lambda \frac{w_t}{\nu} ; t = 1, 2, , , T$$

where λ is the Lagrange multiplier associated with the constraint (19). We assume that h_t is strictly positive. The interpretation of λ is that it is the marginal utility of wealth. In addition to (32) and (33), equations (30) and (31) also belong to the first order conditions.

The model (30)-(33) is often called the λ-constant labor supply model.

Let

(34) $\lambda_t = \left[\frac{1+\rho}{1+r}\right]^t \lambda$

From eqs (30)-(34) we then observe that we get the following solutions for C_t and h_t:

(35) $C_t = C(\lambda_t, w_t)$

(36) $h_t = h(\lambda_t, w_t)$

The solution for λ follows from feeding (35) and (36) into (30) and we observe that λ will depend on initial wealth A_0 and the whole time-path for the wage rate, and of course also the interest rate r and the rate of time preference ρ.

The functions C(.) and h(.) in (35) and (36) are often referred to as the λ-constant consumption and labor supply functions, respectively. We observe that if the utility function is separable in consumption and leisure, then to condition labor supply on λ_t is the same as conditioning on C_t. Hence, in this case, the elasticity of labor supply wrt the wage rate, given λ_t, will be the same as the elasticity of labor supply wrt the wage rate, given consumption. In the preceding section this elasticity was called the Frisch elasticity. If the utility function is the same as specified in (8), then the elasticity of labor supply wrt the wage rate, given consumption, in the life-cycle model is given by the expression in (17).

Inspection of the λ-constant consumption and labor supply functions reveals that consumption and labor supply decisions at a point in time t are related to variables outside the decision period t only through λ. Thus, except for the value of the current wage rate, λ summarizes all information about lifetime wages and property income that a consumer requires to determine his or her optimal current consumption and labor supply.

To demonstrate some results and to obtain some closed form solutions we will simplify exposition by assuming that $r=\rho$ and utility to be log-linear and separable function of consumption and leisure. Instead of (8) we have

(37) $U = a \ln C + b \ln L; \{a, b,\} > 0.$

Moreover we will assume that T is sufficiently large to yield $\sum_{t=1}^{T}\left[\frac{1}{1+r}\right]^t = \frac{1}{r}$

We then get the following solutions for optimal consumption, labor supply and λ:

$$(38)\ C_t = \frac{a}{\lambda}$$

$$(39)\ L_t = \frac{b}{\lambda(w_t/\nu)}$$

$$(40)\ \lambda = \frac{a+b}{rF},$$

where

$$(41)\ F = A_0 + \sum_{\tau=1}^{T}\left[\frac{1}{1+r}\right]^{\tau}\frac{w_\tau}{\nu}$$

We observe that F is the sum of initial wealth and present value of future earning if the individual were working a maximum hours (L_t=0) in every period. F can be interpreted as the total wealth of the individual when he or she enters the labor market in period 1. From (40) we observe that λ is inversely related to the interest income, rF, related to total wealth.

From (38) we get that the optimal consumption path, C_t, is constant over time, that is $C_{t+1}=C_t$. From (39) we get that

$$(42)\quad \frac{L_t}{L_{t+1}} = \frac{w_{t+1}}{w_t}$$

This result is equivalent to what we obtained in the two period case (see eq (28), with γ=0 and r=ρ) and implies that if the wage rate in period t is raised relative to the wage rate in period t+1, hours of work in period t will increase relative to in period t+1.

From (38) and (40) we get

$$(43)\quad C_t = \frac{a}{a+b} rF$$

Thus optimal consumption is proportional to the interest income related to total wealth.

We also easily get that optimal labor supply is given by

$$(44)\ h_t = \frac{1}{\nu}\left[1 - \frac{b}{a+b}\frac{rF}{(w_t/\nu)}\right],$$

and we observe that an increase in w_t, given that the wage rate in all other period are constant, have two effects on labor supply.

First, a higher wage rate, given total wealth F, will have a positive impact on labor supply in the current period t. Second, since total wealth F is also increased, a negative wealth effect on labor supply in the current period t will occur. Given that r is positive, the net effect (given the assumptions made above) will always be positive. We observe that an equal proportional increase in all wage rates will have no impact on labor supply in any period.