ECON 4310
MCandless and Wallace, Chapter 5[1]

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4 Perfect foresight and long-lived bonds

- Purpose of lecture: introduce long-lived bonds in order to understand the role of expectations for the competitive equilibrium.

Definition 1 *A k-period bond is one that is issued in period t and will be paid off one unit of consumption good per bond in period $t+k$.*

- Let $B_k(t)$ denote the number of outstanding k-period bonds at period t. One period later they are called $k-1$-*period bonds* and they are then denoted $B_{k-1}(t+1)$. Since the bonds do not pay any *coupons*, they are called *zero-coupon-bonds* (see definition 3 below).
- Individuals hold $b_k^h(t) \geq 0$ number of k-period bonds.
- Let $p_k(t)$ be the price of a k-period bond at period t.
- Let $p_{k-1}^{h,e}(t+1)$ be the expectation – held by individual h – of the price of a $k-1$ period bond at period $t+1$, where the expectation is formed in period t. Assume *certainty* of expectations – *point expectations*.
- The budget constraint for individual h if k-period bonds and private lending exist is

$$\begin{aligned} c_t^h(t) &\leq \omega_t^h(t) - t_t^h(t) - l^h(t) - p_k(t) b_k^h(t) \\ c_t^h(t+1) &\leq \omega_t^h(t+1) - t_t^h(t+1) + r(t) l^h(t) + p_{k-1}^{h,e}(t+1) b_k^h(t). \end{aligned}$$

- *Expected wealth*: the lifetime budget constraint is then

$$\begin{aligned} c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} &\leq \omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)} \\ &\quad + \frac{b_k^h(t)}{r(t)} \left[p_{k-1}^{h,e}(t+1) - r(t) p_k(t) \right]. \end{aligned}$$

[1]The lecture notes of the first part of the class (first 7-8 lectures) are largely based on McCandless and Wallace. Correspondance to kjetil.storesletten@econ.uio.no

Thus, the demand of individual h for k-period bonds equals

$$b_k^h(t) = \begin{cases} 0 & \text{if } r(t)p_k(t) > p_{k-1}^{h,e}(t+1) \\ \infty & \text{if } r(t)p_k(t) < p_{k-1}^{h,e}(t+1) \\ ? & \text{if } r(t)p_k(t) = p_{k-1}^{h,e}(t+1) \end{cases}$$

Proposition 1 *If there exists unanimity of expectations such that*

$$p_{k-1}^{h,e}(t+1) = p_{k-1}^{e}(t+1),$$

for all h of period t, and if some k-period bonds exist at time t, then in any equilibrium, $r(t)p_k(t) = p_{k-1}^e(t+1)$.

- Unanimity of expectations implies that the lifetime budget constraint for agent h of period t is

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} \leq \omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)},$$

  i.e. only the endowments determine wealth.

4.1 Temporary equilibrium

Definition 2 *Given $\left\{u_t^h(.,.), \omega_t^h, t_t^h, p_{k-1}^e(t+1)\right\}$, a time t "temporary" equilibrium is a pair of prices $[r(t), p_k(t)]$ such that the following equilibrium conditions are fulfilled*

(i) $r(t)p_k(t) = p_{k-1}^e(t+1)$

(ii) $S_t(r(t)) = p_k(t)B_k(t)$

- To find a temporary equilibrium with long-lived bonds, we proceed as before: Find the savings functions of the individuals, aggregate these and solve for prices (using the equilibrium conditions). Finally, compute the implied quantities. Note that with only k-period bonds we have two prices and two equilibrium conditions. Use *(i)* to solve for $p_k(t)$ and obtain

$$S_t(r(t)) = \frac{p_{k-1}^e(t+1)}{r(t)} B_k(t).$$

  The right-hand side is decreasing in $r(t)$. If the left-hand side is increasing in $r(t)$ (savings are non-decreasing in $r(t)$ if consumption in period $t+1$ is a normal good), we know that we can, given expectations, solve for $r(t)$. The expectations carry all the information we need to know about the future.

- Alternatively, we could have solved for $p_k(t)$ instead, i.e.

$$S_t\left(\frac{p_{k-1}^e(t+1)}{p_k(t)}\right) = p_k(t)B_k(t). \tag{1}$$

  Let f_t be the *price function* for government bonds defined implicitly by equation (1), that is,

$$p_k(t) = f_t\left(p_{k-1}^e(t+1), B_k(t)\right).$$

4.2 Perfect foresight

- How are expectations about the future formed? "*Rational expectations*" are expectations which are formed using all relevant (and available) information.

  – Stochastic environment ⇒ agents make errors but are right *on average* (e.g. even sophisticated investors lose money from time to time, but they typically win in the long run).

  – Deterministic world: possible to figure out exactly what will happen (if one has access to all relevant information). *Perfect foresight* is deterministic version of rational expectations.

- Why perfect foresight?

  1. One would lose utility if expectations differ from perfect foresight. Moreover, if someone else has non-rational expectations, one can, with perfect foresight, make arbitrage.
  2. Prevent systematic errors.

- Perfect foresight implies
$$p_k^e(t+1) = p_k(t+1).$$

Proposition 2 *A perfect foresight competitive equilibrium with long-lived bonds is an infinite sequence of prices $p_k(t)$ and $r(t)$ and endogenous variables such that the time t values are a temporary equilibrium satisfying*

$$p_k^e(t+1) = p_k(t+1).$$

- Finding a competitive equilibrium can be difficult. Let us consider a **special case**: Assume the government issues $B_k(t)$ units of k-period bonds in period t, and that no other bonds are ever issued (⇒ finance repayment in period $t+k$ through taxes).

- Solution approach:

  1. Make use of the fact that no other bonds are ever issued (so outstanding bonds are known for each t).
  2. Find $p_{k-j}^e(t+j)$ for $j = k$.
  3. Use the first condition for a temporary equilibrium to determine a relationship between $r(t+j-1)$ and $p_{k-j+1}(t+j-1)$ for $j = k$.
  4. Use the second condition for a temporary equilibrium to determine $r(t+j-1)$ for $j = k$.
  5. Solve for $p_{k-j+1}(t+j-1)$ using the relationship in 3 above for $j = k$.
  6. Repeat steps 2-5 for $j = k-1, k-2, ..., 1$.

STEP 1 If $B_k(t)$ bonds are issued then with no other bonds ever issued we have the same amount of bonds at all k dates:

$$B_k(t) = B_{k-1}(t+1) = B_{k-2}(t+2) = ... = B_1(t+k-1)$$

STEP 2 The one-period bonds are identical to the bonds analyzed in chapter 3 of M&W. The expected (perfect foresight) price in period $t+k$ equals

$$p_0^e(t+k) = p_0(t+k) = 1.$$

STEP 3 From the first condition for a (temporary) equilibrium, we have

$$r(t) = \frac{p_{k-1}^e(t+1)}{p_k(t)}$$

which generalizes to

$$r(t+k-1) = \frac{p_0^e(t+k)}{p_1(t+k-1)} = \frac{1}{p_1(t+k-1)}.$$

STEP 4 We can now use the second condition for a (temporary) equilibrium,

$$S_t(r(t)) = \frac{p_{k-1}^e(t+1)}{r(t)} B_k(t)$$

which generalizes to

$$S_{t+k-1}(r(t+k-1)) = \frac{p_0^e(t+k)}{r(t+k-1)} B_1(t+k-1) = \frac{B_1(t+k-1)}{r(t+k-1)}$$

$\Rightarrow$ one equation and one unknown, $r(t+k-1)$, which we can be solve for.

STEP 5 We then solve for $p_1(t+k-1)$ using condition *(i)*, i.e.

$$p_1(t+k-1) = \frac{1}{r(t+k-1)}.$$

- **Now, repeat this procedure for $j = k-1$:**

STEP 2 again: Under perfect foresight, we know that

$$p_1^e(t+k-1) = p_1(t+k-1)$$

STEP 3 again: From the first condition for a (temporary) equilibrium, we have

$$r(t+k-2) = \frac{p_1^e(t+k-1)}{p_2(t+k-2)} = \frac{p_1(t+k-1)}{p_2(t+k-2)} = \frac{1}{r(t+k-1)\, p_2(t+k-2)}.$$

STEP 4 again: We can again use the second condition for a (temporary) equilibrium,

$$S_{t+k-2}(r(t+k-2)) = p_2(t+k-2)\, B_2(t+k-2) = \frac{B_2(t+k-2)}{r(t+k-1)r(t+k-2)}$$

$\Rightarrow$ one equation and one unknown, $r(t+k-2)$, which we can be solve for.

STEP 5 again: Finally, we solve for $p_2(t+k-2)$ using condition *(i)*.

- Repeat this procedure of iterating backwards until we reach period t, where we have

$$S_t(r(t)) = p_k(t)\, B_k(t) = \frac{B_k(t)}{r(t)r(t+1)...r(t+k-1)} = \frac{B_k(t)}{\Pi_{s=0}^{k-1} r(t+s)} \tag{2}$$

  where $r(t)$ is the only unknown $\Rightarrow$ can solve for it.

- Alternatively, the price sequence could have been found using the pricing function $f_t(.)$. We know the expected price in period $t+k$ equals

$$p_0^e(t+k) = p_0(t+k) = 1$$

  and that $B_k(t) = B_1(t+k-1) \equiv \bar{B}$, hence

$$p_1(t+k-1) = f(p_0^e(t+k), \bar{B}) = f(1, \bar{B}).$$

  Using the same logic in the previous period (together with perfect foresight), we get

$$p_2(t+k-2) = f(p_1^e(t+k-1), \bar{B}) = f(f(1, \bar{B}), \bar{B}).$$

  Repeating the steps 2-5 above for periods $t+k-3, ..., t$, is the same as applying the pricing function repeatedly on $p_0(t+k)$. If the environment differ over time (e.g. changes in endowments), the function differs but the procedure is the same.

4.3 Term structures and interest rates

Definition 3 *A k-period "coupon bond" offers a stream of payments $\left\{x^k(t+i)\right\}$ for i equals 1 to k. The quantity $x^k(t+i)$ is the coupon payment in period $t+i$.*

Definition 4 *The "gross yield" or the "gross internal rate of return" on a k-period coupon bond that offers a stream of payments $\left\{x^k(t+i)\right\}$ for i equals 1 to k and has a price $p_k(t)$ at time t, is defined as the number $r_k(t)$ that satisfies*

$$p_k(t) = \sum_{i=1}^{k} \left[\frac{1}{r_k(t)}\right]^i x^k(t+i)$$

Example 1 *The coupon payments for a zero coupon bond are*

$$x^k(t+i) = \begin{cases} 0 & i = 1, ..., k-1 \\ 1 & i = k \end{cases}$$

The gross internal rate of return then satisfies

$$\begin{aligned} p_k(t) &= \left[\frac{1}{r_k(t)}\right]^k \\ &\Leftrightarrow \\ r_k(t) &= \frac{1}{[p_k(t)]^{1/k}} \end{aligned} \tag{3}$$

Proposition 3 *Under the perfect foresight hypothesis the gross internal rate of return on a k-period zero-coupon bond at time t is a geometric average of the one-period interest rates that will exits during the life of the k-period bond, that is,*

$$r_k(t) = [r(t)r(t+1)...r(t+k-2)r(t+k-1)]^{1/k} \tag{4}$$

Proof: From equation (3) we have for a zero-coupon bond

$$r_k(t) = \left[\frac{1}{p_k(t)}\right]^{1/k}.$$

In deriving the perfect foresight competitive equilibrium for this economy we derived an equilibrium condition (savings market clearing) as equation (2) above:

$$S_t(r(t)) = p_k(t)B_k(t) = \frac{B_k(t)}{\Pi_{s=0}^{k-1} r(t+s)}$$

thus

$$\frac{1}{p_k(t)} = \Pi_{s=0}^{k-1} r(t+s).$$

Combining the two we have

$$r_k(t) = \left[\frac{1}{p_k(t)}\right]^{1/k} = \left[\Pi_{s=0}^{k-1} r(t+s)\right]^{1/k}.$$

QED

- **The term structure**: Suppose now that bonds of different maturities co-exist. At time t these bonds have different prices $p_k(t)$. Using equation (3), we can determine the internal rate or return $r_k(t)$ implied by the price $p_k(t)$ for all different maturities. The implied sequence $r_1(t)$, $r_2(t)$... is called the *term structure of interest rates*. The hypothesis that the sequence of interest rates can be determined by equation (4) is called *the expectations hypothesis of the term structure of interest rates.*

Example 2 *Consider an economy with outstanding bonds maturing in 1,2,3, and 4 periods. Using equation (4), we have*

$$\begin{aligned}
r_1(t) &= [r(t)]^{1/1} \\
r_2(t) &= [r(t)r(t+1)]^{1/2} \\
r_3(t) &= [r(t)r(t+1)r(t+2)]^{1/3} \\
r_4(t) &= [r(t)r(t+1)r(t+2)r(t+3)]^{1/4}
\end{aligned}$$

We can then use the prices $p_1(t)$*,* $p_2(t)$*,* $p_3(t)$*, and* $p_4(t)$*, to calculate all* $r_k(t)$ *up to* $k = 4$*. We then calculate* $r(t)$ *using the first line above,* $r(t+1)$ *using the second line above, etc. We thus have a "theoretical" sequence of* $r(j)$ *based on the expectations hypothesis. To test (a strong version of) this hypothesis, compare the sequence of* $r(j)$ *with the realized data.*

- **Irrelevancy of maturity composition**: Corresponding to any perfect foresight equilibrium in which there are long term bonds, there exists another equilibrium in which the long-term bonds are replaced by a sequence of one-period bonds, but everything else is the same. This is denoted the *irrelevancy of maturity composition.*