A simple Consumption-based Capital Asset Pricing Model

Integrated with McCandless and Wallace

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6 Introduction

- Purpose of lecture: understand the consumption-based capital asset pricing model (CCAPM).
- Questions:
  1. how does risk affect the price of assets and the "equity premium"?
  2. which assets pay a high expected return?

6.1 Environment

- Assume the population is constant, $N(t) = N$ for all $t \geq 1$, and that endowments are identical across individuals and time:

$$\omega_t^h = [\omega_1, \omega_2]$$

- There are two assets in this economy,
  1. private borrowing and lending, paying a risk-free rate of return $r(t)$.
  2. A units of land, yielding an uncertain crop $d(t)$ per unit of land. Assume that the yield in each period t is stochastic:

$$\begin{aligned} d(t) &= d + \varepsilon_t \\ \varepsilon_t &\in \begin{cases} +\sigma & \text{with probability } \frac{1}{2} \\ -\sigma & \text{with probability } \frac{1}{2} \end{cases} \end{aligned}$$

where $\sigma \in [0, d)$. Thus,

$$\begin{aligned} E(\varepsilon_t) &= 0 \\ var(\varepsilon_t) &= \sigma^2 \end{aligned}$$

- Assume preferences are time-separable and identical across individuals and time:

$$u_t^h(c_t^h(t), c_t^h(t+1)) = u(c_t^h(t)) + \beta u(c_t^h(t+1)).$$

The function $u(.)$ is assumed to be differentiable and concave (these properties of u guarantee that u_t^h is convex and differentiable, see lecture 1).

6.2 Solving the individual's problem

- When the return on land is stochastic (and preferences are time-additive), agents maximize expected utility. That is, agents solve

$$\max_{\{a^h(t), l^h(t)\}} \mathrm{E}_t \left\{ u(c_t^h(t)) + \beta u(c_t^h(t+1)) \right\}$$

subject to budget constraints:

$$\begin{aligned} c_t^h(t) &\leq \omega_1 - l^h(t) - p(t)a^h(t) \\ c_t^h(t+1) &\leq \omega_2 + r(t)l^h(t) + (p(t+1) + d + \varepsilon(t+1))\, a^h(t) \end{aligned}$$

Note that the expectation operator E_t is the expectation conditional on information at time t, i.e. the expectation of the stochastic variable $\varepsilon(t+1)$.

- The solution to the individual's problem can be found by substituting the budget constraints into the utility function and differentiating the expected utility with respect to individual demand for lending $l^h(t)$ and land $a^h(t)$

  - For lending $l^h(t)$ we have that

$$\begin{aligned} 0 &= \frac{d}{dl^h(t)} \mathrm{E}_t \left\{ u(c_t^h(t)) + \beta u(c_t^h(t+1)) \right\} \\ &= \mathrm{E}_t \left\{ \frac{\partial u\left(c_t^h(t)\right)}{\partial c_t^h(t)} \frac{\partial c_t^h(t)}{\partial l^h(t)} + \beta \frac{\partial u\left(c_t^h(t+1)\right)}{\partial c_t^h(t+1)} \frac{\partial c_t^h(t+1)}{\partial l^h(t)} \right\} \\ &= \mathrm{E}_t \left\{ -\frac{\partial u\left(c_t^h(t)\right)}{\partial c_t^h(t)} + r\,(t)\,\beta \frac{\partial u\left(c_t^h(t+1)\right)}{\partial c_t^h(t+1)} \right\}. \end{aligned} \quad (1)$$

  - for land $a^h(t)$:

$$\begin{aligned} 0 &= \frac{d}{da^h(t)} \mathrm{E}_t \left\{ u(c_t^h(t)) + \beta u(c_t^h(t+1)) \right\} \\ &= \mathrm{E}_t \left\{ \frac{\partial u\left(c_t^h(t)\right)}{\partial c_t^h(t)} \frac{\partial c_t^h(t)}{\partial a^h(t)} + \beta \frac{\partial u\left(c_t^h(t+1)\right)}{\partial c_t^h(t+1)} \frac{\partial c_t^h(t+1)}{\partial a^h(t)} \right\} \\ &= \mathrm{E}_t \left\{ -p(t) \frac{\partial u\left(c_t^h(t)\right)}{\partial c_t^h(t)} + \beta \left[p(t+1) + d + \varepsilon(t+1)\right] \frac{\partial u\left(c_t^h(t+1)\right)}{\partial c_t^h(t+1)} \right\}. \end{aligned} \quad (2)$$

- Note that in order for the "first order equations" to be sufficient conditions for optimum, there must be no borrowing or shortsale constraints for the agents.
- Since $c_t^h(t)$ is known in period t, any function of $c_t^h(t)$ can be taken outside the expectations sign, and equations (1) and (2) can be rewritten to get what is called the fundamental asset pricing equations:

$$\frac{1}{r(t)} = \mathrm{E}_t\left\{\beta\frac{\frac{\partial u\left(c_t^h(t+1)\right)}{\partial c_t^h(t+1)}}{\frac{\partial u\left(c_t^h(t)\right)}{\partial c_t^h(t)}}\right\} = \mathrm{E}_t\left\{\beta\frac{u'\left(c_t^h(t+1)\right)}{u'\left(c_t^h(t)\right)}\right\} \tag{3}$$

$$\begin{aligned} p(t) &= \mathrm{E}_t\left\{[p(t+1)+d+\varepsilon(t+1)]\,\beta\frac{\frac{\partial u\left(c_t^h(t+1)\right)}{\partial c_t^h(t+1)}}{\frac{\partial u\left(c_t^h(t)\right)}{\partial c_t^h(t)}}\right\} \\ &= \mathrm{E}_t\left\{[p(t+1)+d+\varepsilon(t+1)]\,\beta\frac{u'\left(c_t^h(t+1)\right)}{u'\left(c_t^h(t)\right)}\right\} \end{aligned} \tag{4}$$

  Note also that since equations (3) and (4) must hold for any agent, all agents "agree" on the prices in equilibrium.

6.3 Solving for the competitive equilibrium

- The competitive equilibrium requires (i) individual optimization, and (ii) market clearing (see Definition 1 in lecture 1). Three markets are required to clear for each period t:

  1. the market for private lending, which requires
  $$\sum_{h=1}^{N(t)} l^h(t) = 0.$$
  2. the market for land, which requires
  $$\sum_{h=1}^{N(t)} a^h(t) = A.$$
  3. the goods market, which requires
  $$\sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t)} c_t^h(t+1) = \sum_{h=1}^{N(t)} \omega_t^h(t) + \sum_{h=1}^{N(t)} \omega_t^h(t+1) + d(t)A.$$

- Since all individuals $h \in \{1,2,...,N(t)\}$ in any generation t are identical (same endowments and same preferences), their optimal demand for lending, $l^h(t)$, and land, $a^h(t)$, must be the same for all h. Hence, in any competitive equilibrium it must be that asset demands for all h are given by

$$\begin{aligned} l^h(t) &= 0 && (5) \\ a^h(t) &= \frac{A}{N(t)} = \frac{A}{N}. && (6) \end{aligned}$$

- Claim: market clearing for land and private lending guarantees goods market clearing. Proof: In this case, total consumption for the young and the old in period t are given by

$$\begin{aligned} \sum_{h=1}^{N(t)} c_t^h(t) &= \sum_{h=1}^{N(t)} \left(\omega_1 - p(t)\frac{A}{N(t)}\right) = N\omega_1 - p(t)A \\ \sum_{h=1}^{N(t-1)} c_t^h(t) &= \sum_{h=1}^{N(t-1)} \left(\omega_2 + (p(t) + d + \varepsilon(t)) \frac{A}{N(t-1)}\right) \\ &= N\omega_2 + (p(t) + d + \varepsilon(t)) A \\ &= N\omega_2 + p(t)A + d(t)A \end{aligned}$$

Summing both equations, we get

$$\sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_t^h(t) = N\omega_1 + N\omega_2 + d(t)A,$$

i.e. market clearing in the goods market. QED

- What remains now is to find prices $p(t)$ and $r(t)$ for each t such that asset demands are given by (5) and (6). Since the environment is stationary, we guess on a stationary equilibrium, i.e. $p(t) = p$ and $r(t) = r$ for all $t \geq 1$. Moreover, exploiting the market clearing conditions for $l^h(t)$ and $a^h(t)$ (i.e. equations (5)-(6)), we can rewrite the asset pricing equations (3)-(4) as

$$\begin{aligned} \frac{1}{r} &= \beta \mathrm{E}_t \left\{ \frac{u'(\omega_2 + [p + d + \varepsilon(t+1)] A/N)}{u'(\omega_1 - pA/N)} \right\} && (7) \\ p &= \beta \mathrm{E}_t \left\{ \frac{u'(\omega_2 + [p + d + \varepsilon(t+1)] A/N)}{u'(\omega_1 - pA/N)} [p + d + \varepsilon(t+1)] \right\} && (8) \end{aligned}$$

- Since equations (7) and (8) incorporates individual optimization for all agents, and since they also imply market clearing in all markets, these equations are now our equilibrium conditions.

6.4 Imposing further restrictions

- In order to get sharper results, we need to make some assumptions about preferences, crop, and endowments.

6.4.1 Risk neutral agents

- "Risk neutral" preferences means that the utility function is linear in consumption, so that marginal utility is a constant:

$$
\begin{aligned}
u(c) &= \alpha c \\
u'(c) &= \frac{\partial u(c)}{\partial c} = \alpha
\end{aligned}
$$

- The equilibrium conditions (equations (7) and (8)) then become

$$
\begin{aligned}
\frac{1}{r} &= \beta \mathrm{E}_t \left\{ \frac{\alpha}{\alpha} \right\} = \beta \\
p &= \beta \mathrm{E}_t \left\{ \frac{\alpha}{\alpha} \left[p + d + \varepsilon(t+1) \right] \right\} \\
&= \beta \left[p + d \right] + \beta \mathrm{E}_t \left\{ \varepsilon(t+1) \right\} = \beta \left[p + d \right],
\end{aligned}
$$

which implies the same relation between price of land and the interest rate on private lending, namely

$$
r = \frac{p+d}{p} = \frac{1}{\beta}.
$$

- One useful alternative way of expressing the differences between the risky asset and the safe asset is to consider the differences in terms of expected return. Let $\hat{r}$ denote the expected risk premium, i.e. the expected return on land minus the return on lending:[1]

$$
\begin{aligned}
\hat{r} &= \mathrm{E}_t \left\{ \frac{p + d + \varepsilon(t+1)}{p} - r \right\} \\
&= \frac{p+d}{p} - r + \mathrm{E}_t \left\{ \frac{\varepsilon(t+1)}{p} \right\} = 0
\end{aligned}
$$

- Bottom line: uncertainty don't matter because agents are indifferent to risk (risk neutral).

6.4.2 Risk averse agents

- A more interesting case is when agents are risk averse (i.e. dislike risk). In order to solve this case in a simple way, we make two restrictions

  1. Preferences are assumed to be logarithmic, i.e.:

$$
\begin{aligned}
u(c) &= \log(c) \\
&\Rightarrow \\
u'(c) &= \frac{1}{c}.
\end{aligned}
$$

[1] This is also labeled the *expected excess return* on land relative to the safe asset.

Given our notation, this means that u_t^h is given by

$$u_t^h\left(c_t^h(t), c_t^h(t+1)\right) = \log c_t^h(t) + \beta \log c_t^h(t+1)$$

2. Endowments are assumed to be $\omega_t^h = [\omega_1, 0]$ for all agents.

- Given restrictions 1 and 2, the second equilibrium condition (8) simplifies to

$$\begin{aligned} p &= \beta \mathrm{E}_t \left\{ \frac{\omega_1 - pA/N}{[p + d + \varepsilon(t+1)]\, A/N} [p + d + \varepsilon(t+1)] \right\} \\ &= \beta \left(\frac{N}{A}\omega_1 - p \right), \end{aligned}$$

which yields $p = \frac{\beta}{1+\beta}\frac{N}{A}\omega_1$. The first equilibrium condition (7) then simplifies to

$$\begin{aligned} \frac{1}{r} &= \beta \mathrm{E}_t \left\{ \frac{\omega_1 - pA/N}{[p + d + \varepsilon(t+1)]\, A/N} \right\} \\ &= \beta \mathrm{E}_t \left\{ \frac{\frac{1}{1+\beta}\omega_1}{\frac{\beta}{1+\beta}\omega_1 + (d + \varepsilon(t+1))\, A/N} \right\} \\ &= \frac{\beta}{2} \frac{\frac{1}{1+\beta}\omega_1}{\frac{\beta}{1+\beta}\omega_1 + (d+\sigma)\, A/N} + \frac{\beta}{2} \frac{\frac{1}{1+\beta}\omega_1}{\frac{\beta}{1+\beta}\omega_1 + (d-\sigma)\, A/N} \\ &= \beta \frac{\beta + \frac{1+\beta}{\omega_1} dA/N}{\left(\beta + \frac{1+\beta}{\omega_1}\frac{A}{N} d\right)^2 - \left(\frac{1+\beta}{\omega_1}\frac{A}{N}\right)^2 \sigma^2}, \end{aligned} \tag{9}$$

which yields

$$\begin{aligned} r &= 1 + \frac{1+\beta}{\beta\omega_1}\frac{A}{N} d - \frac{\frac{1+\beta}{\beta\omega_1}\frac{A}{N}}{\frac{\beta}{1+\beta}\frac{\omega_1}{A/N} + d}\sigma^2 \\ &= \frac{p+d}{p} - \frac{\sigma^2}{(p+d)\, p}. \end{aligned}$$

Note that in order to arrive at equation (9), we used the definition of the stochastic variable $\varepsilon(t+1)$ from section 6.1 above.

- Note the following facts:

1. There can only be one stationary equilibrium.
2. If land is risk-free, i.e. $\sigma = 0$, then
   - The return on land equals the return on lending, $r = \frac{p+d}{p}$, as in the risk neutral case.
   - The price of land equals the discounted value of the future endowments, $p = \frac{d}{r-1}$.

3. The only aspects of the risky asset that matter for prices are the expected crop, d, and the variance of the crop, σ^2.
4. The expected equity premium is now

$$\begin{aligned}
\hat{r} &= \mathrm{E}_t \left\{ \frac{p + d + \varepsilon(t+1)}{p} - r \right\} \\
&= \mathrm{E}_t \left\{ \frac{p + d + \varepsilon(t+1)}{p} - \frac{p+d}{p} + \frac{\sigma^2}{(p+d)\,p} \right\} \\
&= \frac{\sigma^2}{(p+d)\,p} + \mathrm{E}_t \left\{ \frac{\varepsilon(t+1)}{p} \right\} \\
&= \frac{\frac{1+\beta}{\omega_1}\frac{A}{N}}{\frac{\beta}{1+\beta}\frac{\omega_1}{A/N} + d}\sigma^2
\end{aligned}$$

Bottom line, the equity premium is decreasing in d and increasing in σ^2.

6.5 What assets get high expected return?

- Suppose we introduce a new asset in the economy, a tree yielding a crop $f(t)$ per tree. Assume that the yield in each period t is stochastic:

$$\begin{aligned}
f(t) &= d + \delta_t \\
\delta_t &\in \begin{cases} +s & \text{with probability } \frac{1}{2} \\ -s & \text{with probability } \frac{1}{2} \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
E\left(f\left(t\right)\right) &= d = E\left(d\left(t\right)\right) \\
E(\delta_t) &= 0 \\
var(\delta_t) &= s^2
\end{aligned}$$

Moreover, assume that the yield on the tree is correlated with the yield on the land,

$$corr\left(\delta_t, \varepsilon_t\right) = M.$$

- To facilitate notation, define

$$m_{t+1} \equiv \beta \frac{\frac{\partial u\left(c_t^h(t+1)\right)}{\partial c_t^h(t+1)}}{\frac{\partial u\left(c_t^h(t)\right)}{\partial c_t^h(t)}} = \beta \frac{u'\left(c_t^h(t+1)\right)}{u'\left(c_t^h(t)\right)} = \frac{1}{MRS_{t,t+1}}$$

The asset pricing equations then become

$$\begin{aligned}
\frac{1}{r\left(t\right)} &= \mathrm{E}_t \left\{ m_{t+1} \right\} &\quad (10) \\
p(t) &= \mathrm{E}_t \left\{ \left[p(t+1) + d + \varepsilon(t+1) \right] m_{t+1} \right\}
\end{aligned}$$

Using the asset pricing equation (4), the price of the tree, $p^T(t)$, can be computed as

$$\begin{aligned} p^T(t) &= \mathrm{E}_t\left\{\left[p^T(t+1)+d+\delta(t+1)\right]m_{t+1}\right\} \\ &\Rightarrow \\ 1 &= \mathrm{E}_t\left\{R^T_{t+1}m_{t+1}\right\} \end{aligned}$$

where

$$R^T_{t+1} \equiv \frac{p^T(t+1)+d+\delta(t+1)}{p^T(t)}$$

is the return on the tree in period $t+1$.

- Using the definition

$$cov(x,y) = E(x\cdot y) - E(x)\cdot E(y),$$

we have

$$\begin{aligned} 1 &= \mathrm{E}_t\left\{R^T_{t+1}m_{t+1}\right\} \\ &= cov_t\left(R^T_{t+1},m_{t+1}\right) + \mathrm{E}_t\left\{R^T_{t+1}\right\}\cdot\mathrm{E}_t\left\{m_{t+1}\right\} \end{aligned}$$

Rewriting and using equation (10), we get

$$\begin{aligned} \mathrm{E}_t\left\{R^T_{t+1}\right\} &= \frac{1}{\mathrm{E}_t\left\{m_{t+1}\right\}} - \frac{cov_t\left(R^T_{t+1},m_{t+1}\right)}{\mathrm{E}_t\left\{m_{t+1}\right\}} \\ &= r(t) - \frac{cov_t\left(R^T_{t+1},m_{t+1}\right)}{\mathrm{E}_t\left\{m_{t+1}\right\}} \end{aligned}$$

Substituting in m_{t+1}, we have

$$\begin{aligned} \mathrm{E}_t\left\{R^T_{t+1}\right\} &= r(t) - \frac{cov_t\left(R^T_{t+1},u'\left(c_t(t+1)\right)\right)}{\mathrm{E}_t\left\{u'\left(c_t(t+1)\right)\right\}} \\ &= r(t) - std\left(R^T_{t+1}\right)\cdot corr\left(R^T_{t+1},u'\left(c_t(t+1)\right)\right)\cdot\frac{std\left(u'\left(c_t(t+1)\right)\right)}{\mathrm{E}_t\left\{u'\left(c_t(t+1)\right)\right\}} \quad (11) \end{aligned}$$

Note that the term $u'(c_t(t))$ cancels because it can be taken outside of the conditional expectations terms and so it appears in both the denominator and the enumerator, i.e. if k is a constant and x and y are random variables, then

$$\frac{cov\left(\frac{x}{k},y\right)}{E\left(\frac{x}{k}\right)} = \frac{\frac{1}{k}cov(x,y)}{\frac{1}{k}E(x)} = \frac{cov(x,y)}{E(x)}.$$

- What do we lean from equation (11)?

  1. The risk premium on an asset is linear in $std\left(R^T_{t+1}\right)$.

2. An asset with high variance $var\left(R_{t+1}^{T}\right)$ but zero correlation with $u'\left(c_t(t+1)\right)$ gets no premium over the risk free rate. The only reason an asset gets a positive or negative risk premium (i.e. an expected return larger or smaller than $r\left(t\right)$) is that it provides (positive or negative) insurance against consumption fluctuations.

3. An asset which provides a low return when $u'\left(c_t(t+1)\right)$ is high has a negative correlation,
$$corr\left(R_{t+1}^{T}, u'\left(c_t(t+1)\right)\right) < 0,$$
and therefore gets a high expected return. If the utility function is concave (agents are risk averse, and preferences are convex, as defined in lecture 1), then $u'\left(c\right)$ is high when c is low. Thus, this asset pays a low return when consumption is low. No investors would want to hold this asset if the expected return was equal to the risk-free rate! Thus, it must have a positive risk premium in order to induce people to hold it.

4. Conversely, an asset with positive correlation with $u'\left(c_t(t+1)\right)$ has negative correlation with $c_t(t+1)$ and therefore helps to smooth consumption. People would therefore like to hold it, even if the expected return is lower than the riskfree rate. Insurance is an example of an asset with these characteristics; i.e., it pays off only when consumption is low, and it has a negative expected return.

5. The risk premium is increasing in $std\left(u'\left(c_t(t+1)\right)\right)$. Thus, if either $c_t(t+1)$ is very variable or the marginal utility $u'\left(c_t(t+1)\right)$ is very steep (i.e. the utility function u is very concave), then the risk premium becomes big.