Uncertain investments under limited diversification

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1 Introduction

This paper[1] considers a theory of how individual agents evaluate uncertain investments. It is also shown how it is possible to derive relations characterizing the economy as a whole, such as the one known as the capital asset pricing model (CAPM).[2] The derivation is an alternative to the traditional textbook presentation. This paper does not cover social evaluation of uncertain investments, but such evaluations will usually rely on the same kind of theory which is covered here.[3]

The model below is different from the standard CAPM in the limitations to diversification. In the CAPM each agent's uncertain future wealth is the future value of a portfolio composed optimally by the agent. Each source of uncertain income is a holding of a security, such as a share. The agent considers the probability distribution of the future share value (including dividends, if any) as exogenous, but decides herself how many shares to own. In the model of this paper, however, the agents are not capable of freely composing their portfolios. One or more sources of uncertain income is exogenously given.

In the literature on this, "human capital" is the best-known example on this kind of exogenous source of uncertainty. The human capital is the present value of income from future work. One may invest in this through education.

The reason why an agent cannot easily choose an optimal amount of human capital, is the difficulty in selling parts of it. Of course knowledge and abilities deteriorate, but this does not correspond to selling parts of the capital as in a portfolio model. If human capital were to fit into a standard portfolio model, one would need the price of buying and selling at the same point in time to be the same. The sale of human capital would mean the sale of claims to future income from one's own work. But slavery is prohibited. Moreover, slaves have poor incentives to work, and their effort is hard to monitor. Those with the poorest prospects for working productively would have the strongest incentives to sell. All these problems are well-known from the literature on asymmetric information, known as "moral hazard" and "adverse selection."

In practice most people will thus be left with the uncertainty connected to own future income from work. This is a deviation from the assumptions of the CAPM. In the literature this was recognized early, and one asked how the existence of non-marketable capital would change the equation characterizing the equilibrium in the CAPM. This is treated by Mayers (1972).[4] The model below builds on Mayers, but is more general by using a lemma of Stein (1973). The CAPM appears as a special case when all capital is marketable.

The new result presented below, is the criterion for investing in more of a non-marketable asset. Even when an asset is non-marketable, one may wish to invest in more of it if one can buy it cheaply. This is relevant for investment in education, but other examples may be more directly subject to standard economic evaluation. In particular, I have been concerned with investment in Norwegian petroleum activity. Bøhren and Ekern (1987) discuss the same problem with closely

[1]Thanks to Kåre N. Edvardsen and an anonymous referee for useful comments. The responsibility for remaining errors and omissions is my own.

[2]The paper does not give a complete account of the CAPM, developed by Sharpe (1964), Lintner (1965), and Mossin (1966). At this point the paper is mainly meant as a supplement to existing literature. A thorough review of recent literature on the CAPM and related models is Constantinides (1989).

[3]See, e.g., Sandmo (1972) and Lommerud (1984). A comment on Sandmo's model is in Lund (1988). A further discussion is in Lund (1990) and Lund (1993).

[4]A model closely related to Mayers' is found in Brito (1977). Somewhat less closely related models are discussed in section VIII in Constantinides (1989).

related methods. Their conclusions are also closely related to mine. Their formal derivation is quite brief. It differs from the model below by considering a non-marginal increase in a stock.

For sake of the argument we may consider Norway as a single agent. Like many other petroleum producing countries, Norway has not made significant attempts at selling resources before they are extracted. One should ask why this has not happened. Many motives may have been part of the political process. One important reason why resources seldom are sold in situ, is probably that potential buyers face what is known as political risk: After a sale of resources in situ Norway may be tempted to increase taxes, impose stricter regulations, or nationalize the activities.[5] Such a political risk is another example of moral hazard being an obstacle to diversification.

Political risk is not explicitly part of the model below, but we shall see that the model may be used to describe the investment problem of a resource rich nation.

2 Investment evaluated by a single agent

We set up a two-period model for maximization of expected utility by a single agent. The agent has intital wealth, W_0, at the beginning of period 0, to be distributed between consumption in period 0, C_0, and a number, $n+1$, of assets:

$$W_0 = C_0 + \sum_{j=0}^{n} X_j P_{j0}, \tag{1}$$

where X_j is the quantity of asset j, while P_{jt} is this asset's price in period t. All variables in (1) are non-stochastic (deterministic).

In period 1 consumption equals the total value in period 1 of the assets, plus the value of a number, $k-n$, of other assets, of which the agent has exogenously given amounts.

$$C_1 = \sum_{j=0}^{k} X_j P_{j1}, \tag{2}$$

where the assets with exogenous amounts are numbered $n+1, \ldots, k$. Prices $P_{11}, \ldots, P_{k1}$ are uncertain, i.e., viewed from period 0 they are stochastic variables. Thus also C_1 is stochastic. Asset 0, on the other hand, is assumed to have a deterministic future price, so that $r_0 \equiv (P_{01} - P_{00})/P_{00}$ can be seen as a riskless interest rate. Some of the assets may be real investments, but if these belong to the $n+1$ first, is is assumed for simplicity that they have constant returns to scale.

The agent is assumed to have a time-additive von Neumann and Morgenstern utility function

$$U(C_0, C_1) \equiv u(C_0) + \theta E[u(C_1)], \tag{3}$$

[5] It would require a more complicated model if we should treat political risk in a satisfactory way. Political risk is created by one of the agents, while price uncertainty in the model is conceived as originating outside the model. Political risk implies, nevertheless, that the buyer's willingness to pay is reduced, in the same way as price uncertainty normally does. Resource rich nations wishing to sell resources in situ will therefore often try to commit to not imposing adverse political measures on the buyer afterwards. If such a commitment is credible, the willingness to pay will increase.

where u may be called a per-period utility function, θ is a utility discount factor, and E denotes an expectation.[6] The agent chooses $X_0, \ldots, X_n$ and achieves the maximum utility

$$U^*(W_0, X_{n+1}, \ldots, X_k) \equiv \max_{X_0,\ldots,X_n} U(C_0, C_1) \text{ s.t. } W_0, X_{n+1}, \ldots, X_k, \text{(1), and (2).} \tag{4}$$

Assume now that the per-period utility function u is increasing and strictly concave, and that the P_{j1}'s are not perfectly correlated.[7] Then it is well known that such a risk averse agent will benefit from diversification. The optimal portfolio will usually be divided across investments in most assets. But as long as we consider a single agent, there is nothing to prevent the optimal holding of some assets from being zero or negative.[8] When the budget for C_1 consists of exogenous, uncertain sources of income in addition to those being chosen freely, the characteristics of these $k-n$ last objects are likely to affect the optimal composition of the holdings of the $n+1$ first. This will emerge from the conditions to be derived below.

By combining (1), (2), (3), and (4) we find the maximization problem

$$U^*(W_0, X_{n+1}, \ldots, X_k) = \max_{X_0,\ldots,X_n} \left\{ u\left(W_0 - \sum_{j=0}^{n} X_j P_{j0}\right) + \theta E\left[u\left(\sum_{j=0}^{k} X_j P_{j1}\right)\right]\right\}, \tag{5}$$

with the first order conditions, for $j = 0, \ldots, n$,

$$u'(C_0)(-P_{j0}) + \theta E[u'(C_1) P_{j1}] = 0. \tag{6}$$

This may be rewritten as

$$u'(C_0) P_{j0} = \theta E[u'(C_1)] E(P_{j1}) + \theta \operatorname{cov}[u'(C_1), P_{j1}]. \tag{7}$$

Let now $R_j \equiv 1 + r_j \equiv P_{j1}/P_{j0}$. We call this magnitude the return on asset j, or one plus the rate of return on asset j. Consider first asset 0, having a risk free return. For $j = 0$ the covariance will be zero, and we find

$$R_0 = \frac{u'(C_0)}{\theta E[u'(C_1)]}. \tag{8}$$

The right-hand side of this equation is a marginal rate of substitution between consumption in the periods 0 and 1. As long as it is possible to save or borrow any amount at the risk free rate r_0, then $1 + r_0$ will in optimum be equal to this marginal rate of substitution. Since the agent regards R_0 as exogenous, it is C_0 and C_1 (through $X_0, \ldots, X_n$) which must be adjusted until the equation is satisfied.

By dividing (7) by $u'(C_0)$ and using (8) we may derive the following equation for a risky asset j (for $j = 1, \ldots, n$):

$$P_{j0} = \frac{1}{R_0}\left\{E(P_{j1}) + \frac{\operatorname{cov}[u'(C_1), P_{j1}]}{E[u'(C_1)]}\right\}. \tag{9}$$

[6] C_1 is stochastic. In the way $U(C_0, C_1)$ is defined here, it is not a function of two real variables, but a function of a real and a stochastic variable. Since $E[u(C_1)]$ is a non-stochastic property of the probability distribution of $u(C_1)$, $U(C_0, C_1)$ will be non-stochastic by its definition.

[7] We shall furthermore assume that it is impossible to compose a linear combination of the P_{j1}'s, $j = 1, \ldots, k$, with a variance of zero.

[8] We shall not go into conditions for optimal holdings of all assets to be positive. It is assumed that the maximization problem has a unique solution with all X_j's being finite. This is not obvious: If an asset is first-order stochastically dominated by a linear combination of other assets, the demand for that combination would be infinite, while the demand for the dominated asset would be minus infinity.

The interpretation of (9) is that in optimum, the price of asset j will be equal to a risk adjusted present value of the future uncertain P_{j1}. The present value is obtained through the factor $1/R_0$. Within the curly brackets we find a certainty equivalent for P_{j1}. The risk correction consists in the covariance term. This reflects to what extent asset j contributes to the uncertainty in C_1, or more precisely, to the uncertainty in $u(C_1)$ at the margin.

Since u' is a decreasing function, high values of C_1 will occur together with low values of of $u'(C_1)$. One might be tempted to assume that

$$\operatorname{cov}(C_1, P_{j1}) > 0 \Longleftrightarrow \operatorname{cov}[u'(C_1), P_{j1}] < 0. \tag{10}$$

There are, however, particular probability distributions for which this is not true.[9] In the interpretation of (9) it is nevertheless common to assume that (10) holds.

We have $\operatorname{cov}(C_1, P_{j1}) = \sum_{i=1}^{k} X_i \operatorname{cov}(P_{i1}, P_{j1})$. The contribution from asset j to the uncertainty in C_1 is thus partly due to $X_j P_{j1}$ being included in the budget for C_1, and partly in the covariance with the other uncertain prices. Observe in particular that $\operatorname{cov}(C_1, P_{j1})$ is bound to increase if X_j increases. If we restrict our attention to positive X_j's and X_i's, a large X_j and positive covariances between P_{j1} and the other prices will contribute to $\operatorname{cov}(C_1, P_{j1}) > 0$, so that the risk correction in (9) becomes negative. If the covariances with the other prices are negative, and X_j is relatively small, the risk correction may be positive. In that case one can view asset j as insurance against the uncertainty in C_1, and the required expected return is thus lower than R_0.

Another way of expressing (9) is found by multiplying both sides with R_0/P_{j0} and rearranging, to find

$$E(R_j) - R_0 = \frac{-\operatorname{cov}[u'(C_1), R_j]}{E[u'(C_1)]}. \tag{11}$$

This is called the required expected excess return, in excess of the risk free return. The CAPM is usually expressed in such terms, including a covariance term on the right-hand side of the equation. But in addition it will relate the required expected return to the expected return on the market portfolio. In section 4 we shall see how this portfolio is introduced into the model.

We shall now consider more closely the possibility of investing in more of the assets $n+1, \ldots, k$, which at the outset are available in fixed, exogenous quantities. We may use the solution of (5) to find the value of small changes in such an exogenous quantity. We assume that an opportunity arises to invest an amount I, thereby acquiring one unit more of asset j, where $j \in \{n+1, \ldots, k\}$.

This might, e.g., be an investment in tangible capital or in education. An example is a petroleum producing nation, as mentioned above. During development of an oil field, it is decided how many wells to be drilled, and what extraction capacity to install. This determines how much oil is extracted, and after the natural pressure is lost, the remaining oil is often lost for economic purposes. By investing in more capacity, the nation will thus in an economic sense increase its stock of oil in the period until it is extracted. If the nation's total stock of oil is not optimally chosen at the outset, which it usually isn't, our model is useful for deriving an investment criterion.

From (1), (2), and (3) we see that the increase in utility when X_j (with $j \in \{n+1, \ldots, k\}$) is increased by one unit, will be $\theta E[u'(C_1)P_{j1}]$, while the decrease in utility from reducing the

[9] A counterexample to (10) is the following: Assume four equiprobable states, assume that the variable P takes on the values $(11, 4, 4, 13)$ in these states, while C takes on the corresponding values $(1, 2, 3, 4)$. If $u(C) \equiv \ln(C)$, then $\operatorname{cov}(C, P)$ and $\operatorname{cov}[u'(C), P]$ are both positive. As a curiosity we mention that the book of Huang and Litzenberger (1988) originally is written as if (10) holds, but that the error is corrected in the errata list of the third edition. The inequality F.9 of chapter 19 in Gravelle and Rees (1992) also assumes erroneously that (10) holds.

budget in period 0 by I will be $u'(C_0)I$. Using the envelope theorem on (5) we find that the condition for

$$\frac{\partial U^*}{\partial X_j} - \frac{\partial U^*}{\partial W_0} I \geq 0 \tag{12}$$

(with $j \in \{n+1, \ldots, k\}$) is exactly that

$$u'(C_0)I \leq \theta E[u'(C_1)P_{j1}]. \tag{13}$$

Observe that this investment criterion is valid even in a case where the agent for some reason is unable to choose the holding of any asset freely, i.e., $n = -1$. But assume now that she at least may choose the risk free asset optimally. The equation (8) holds, and $n \geq 0$. Then (13) may be rewritten (just as we derived (9) from (7)):

$$I \leq \frac{1}{R_0} \left\{ E(P_{j1}) + \frac{\text{cov}[u'(C_1), P_{j1}]}{E[u'(C_1)]} \right\}. \tag{14}$$

The left-hand side is the investment cost. The right-hand side is what the agent is maximally willing to pay for the investment. This is the same expression as the right-hand side of (9).

The example of Norway's petroleum wealth may illustrate the meaning of (14). It may appear that political risk or other reasons prevent Norway from selling petroleum in situ. When a high share of petroleum in the national portfolio is maintained in this way, this contributes to reduce the willingness to pay for additional petroleum income. There is reason to believe that the right-hand side in (14) is lower than the price which could have been obtained for petroleum reserves internationally. One has not been able to reduce the national portfolio of petroleum until (9) was satisfied. Nevertheless there may well exist real investment opportunities with an I low enough for (14) to be fulfilled, e.g., within an existing plan for a petroleum field development.

A problem in applications of the investment criterion (14) is that the agent's utility function appears. On one hand this is intuitively reasonable, since risk aversion may have an affect on the willingness to pay for an uncertain source of income. But the model is difficult to apply when the criterion is not expressed in terms of observables. We shall see below that under some conditions, this problem may be alleviated.

3 Simplification: Stein's Lemma

In the previous section we saw that the interpretation of the risk correction had a problem: Even if $u'(C_1)$ is strictly decreasing everywhere, we cannot be certain that $\text{cov}[u'(C_1), P_{j1}]$ has the opposite sign of $\text{cov}(C_1, P_{j1})$. It is interesting to know special cases where this result holds.

In addition we shall in the next section aggregate the model for the whole economy, by summing over agents and assets. In order to arrive at simple expressions, it will be necessary to make simplifying assumptions. These will also give formulae expressed in observable variables.

The CAPM assumes that agents only care about two characteristics of their wealths at the end of the period: The mean and the variance. (The end-of-period wealth is in our model equal to the budget being used for C_1.) When we assume that agents maximize expected utility, one of the following two assumptions will be sufficient for them to care about mean and variance only:

(A1) Their utility functions must be quadratic, i.e., of the form $u(C_1) = a_0 + a_1 C_1 - \frac{1}{2} a_2 {C_1}^2$, where $a_1 > 0$ and $a_2 > 0$.

(A2) The returns must be jointly normally distributed.

Unfortunately none of the assumptions are unproblematic: The first leads to absolute risk aversion being increasing, and that the marginal utility of C_1 becomes negative for large C_1. The second implies that negative returns are possible, which is not true for shares due to limited liability. We shall nevertheless, like much of the literature, see these assumptions as interesting means of simplification. In particular the latter is viewed as a reasonable approximation to empirical data for shares.

If the utility function is quadratic, the marginal utility is linear: $u'(C_1) = a_1 - a_2C_1$. Then we have

$$\operatorname{cov}[u'(C_1), P_{j1}] = -a_2 \operatorname{cov}(C_1, P_{j1}), \tag{15}$$

which solves the problem of interpretation in the previous section: Now it is obvious that these two covariances have opposite signs.

If the returns are normally distributed, we may use a Lemma shown by Stein (1973) and Rubinstein (1976): When g is a differentiable function and X and Y are (jointly) normally distributed,[10] we have

$$\operatorname{cov}[g(X), Y] = E[g'(X)] \operatorname{cov}(X, Y). \tag{16}$$

This means that

$$\operatorname{cov}[u'(C_1), P_{j1}] = E[u''(C_1)] \operatorname{cov}(C_1, P_{j1}). \tag{17}$$

We observe that (15) is a special case of (17). The latter expression is not as simple as (15), but the factor in front of the covariance on the right-hand side is still negative and the same for all j. It turns out that this is sufficient for the derivation in the next section.[11]

4 Aggregation, the CAPM

We consider an economy with H agents, all behaving as assumed in section 2. They may have different values for W_0 and the X_j's, and different utility functions. We introduce a superscript h to denote agent number h.

Assume furthermore that (A1) and/or (A2) hold, so that (17) holds. Equation (11) may be rewritten as

$$E(R_j) - R_0 = \frac{-E[u^{h''}(C_1^h)]}{E[u^{h'}(C_1^h)]} \operatorname{cov}(C_1^h, R_j), \tag{18}$$

for $h = 1, \ldots, H$ and $j = 1, \ldots, n$.

Let R_m be a weigted average of those R_j's for which this equation hold,

$$R_m \equiv \sum_{j=1}^{n} w_j R_j \text{ where } \sum_{j=1}^{n} w_j = 1.$$

At this point we need not specifiy the weights w_j, but R_m will turn out to be the return on the market portfolio. From (18) it is now clear that for $h = 1, \ldots, H$,

$$\sum_{j=1}^{n} w_j[E(R_j) - R_0] = \frac{-E[u^{h''}(C_1^h)]}{E[u^{h'}(C_1^h)]} \sum_{j=1}^{n} w_j \operatorname{cov}(C_1^h, R_j),$$

[10] It is also required that g' is bounded, or a slightly less strict condition, cf. Rubinstein (1976).

[11] Equation (17) is not necessary to arrive at the CAPM. What is necessary is discussed in Ross (1978).

which implies

$$E(R_m) - R_0 = \frac{-E[u^{h''}(C_1^h)]}{E[u^{h'}(C_1^h)]} \operatorname{cov}(C_1^h, R_m). \tag{19}$$

This was the first step in the aggregation. There are still factors in the equation which depend on h. Define now

$$V_x \equiv \sum_{h=1}^{H} \sum_{j=n+1}^{k} X_j^h P_{j1}, \tag{20}$$

$$V_m \equiv \sum_{h=1}^{H} \sum_{j=1}^{n} X_j^h P_{j0}, \tag{21}$$

and, for $j = 1, \ldots, n$,

$$w_j \equiv \frac{P_{j0} \sum_{h=1}^{H} X_j^h}{V_m}. \tag{22}$$

Here V_x is the total stochastic value in period 1 of the $k - n$ last assets, V_m is the total non-stochastic value in period 0 of the n first risky assets, while the w_j's are the weigts of the latter in the economy's total portfolio of these n assets. (It can be verified that these weights sum to 1.) This portfolio may be called the market portfolio — a name well known from the CAPM. In our model it is necessary to make clear that it only includes the marketable risky assets, not the last $k - n$.

By summing (2) for all agents we find

$$\sum_{h=1}^{H} C_1^h = \sum_{h=1}^{H} X_0^h P_{01} + R_m V_m + V_x. \tag{23}$$

This will be useful shortly.

We shall now aggregate (18) and (19) over all agents. Together the two equations imply that

$$\frac{\operatorname{cov}(C_1^h, R_j)}{E(R_j) - R_0} = \frac{E[u^{h'}(C_1^h)]}{-E[u^{h''}(C_1^h)]} = \frac{\operatorname{cov}(C_1^h, R_m)}{E(R_m) - R_0}. \tag{24}$$

This implies, for $j = 1, \ldots, n$,

$$\sum_{h=1}^{H} \frac{\operatorname{cov}(C_1^h, R_j)}{E(R_j) - R_0} = \sum_{h=1}^{H} \frac{\operatorname{cov}(C_1^h, R_m)}{E(R_m) - R_0}. \tag{25}$$

Observe that the denominators do not depend on h. Both on the left and the right-hand side of (25) it is thus sufficient to sum the numerators. For C_1^h we substitute from (23), and find, for $j = 1, \ldots, n$,

$$E(R_j) - R_0 = \frac{V_m \operatorname{cov}(R_j, R_m) + \operatorname{cov}(R_j, V_x)}{V_m \operatorname{var}(R_m) + \operatorname{cov}(R_m, V_x)} [E(R_m) - R_0]. \tag{26}$$

This is Mayer's extension of the CAPM. The equation expresses by how much the requirement of expected return on freely marketable assets, will exceed R_0. The fraction expresses by how much asset j contributes to the total variation in the market portfolio and in V_x, in relative terms.

Observe that the right-hand side of (26) does not depend on individual variables, only on aggregates. This means that, first, the valuation of (and additional unit of) asset j (for $j = 1, \ldots, n$) will be equal for all agents. This is not surprising for the marketable assets: The equality is a necessary condition for market equilibrium. More surprising, perhaps, is the simple form of aggregation: It turns out that only the aggregate magnitudes appear in the model. The distribution of the W_0^h's and the X_j^h's across agents does not matter, and the individual utility functions do not matter. This property of the model is due to the simplification by Stein's Lemma.

The CAPM,

$$E(R_j) - R_0 = \frac{\mathrm{cov}(R_j, R_m)}{\mathrm{var}(R_m)}[E(R_m) - R_0], \tag{27}$$

appears as a special case of (26) when the last terms in both the numerator and in the denominator vanish. This follows if V_x is zero (with certainty), i.e., if the freely marketable assets are the only sources of uncertain income. This is the standard assumption in the CAPM. But also if V_x is uncorrelated with R_m, equation (27) holds for those R_j which are uncorrelated with V_x. Here only the covariances with the aggregate V_x count — it does not matter whether R_m or R_j are correlated with any single agent's exogenous income.

We shall not go in more detail on interpretations or other extensions of the CAPM, about which an extensive literature exists. The model may be extended in various ways to include more than two time periods.[12] Such an extension may also include non-marketable assets, cf. the appendix of Lund (1990).

For the non-marketable assets we had a criterion for investment, the inequality (14). We could not expect this to be equal across agents: Those with a large exogenous X_j^h will, ceteris paribus, have a lower willingness to pay for increasing X_j^h by one unit than other agents. But under (A1) or (A2) the criterion may be simplified considerably (by using (17) and (19)):

$$I \leq \frac{1}{R_0}\left\{E(P_{j1}) - \frac{\mathrm{cov}(C_1^h, P_{j1})}{\mathrm{cov}(C_1^h, R_m)}[E(R_m) - R_0]\right\}. \tag{28}$$

The criterion for investing in more of asset j does no longer depend on the individual utility function, as in (14). The right-hand side of (28) consists of variables which are observable, either on an aggregate or on an individual level.

This result makes precise a statement found in Bøhren and Ekern (1987), p. 25:

> The connection between relevant risk and value may in principle require information both on preferences and the reference portfolio. If the decision maker has access to *capital markets,* no information is needed about preferences. If also the so-called separation property holds, the project value does not depend on the owner's individual characteristics.

(My translation.) We have seen that two assumptions together were sufficient to go from (14) to (28), so that preferences were removed: The individual chooses the holding of *some* assets optimally, so that (19) holds, and (17) holds, which may rely on (A1) or (A2). But "individual characteristics" are not removed, since C_1^h is included in (28).

A main point in Bøhren and Ekern is that quantity uncertainty in a project is only relevant when the project makes up a large share of the decision maker's portfolio. This may be formalized as follows, from (28): Assume a real investment I gives an income pq, where both p, price, and q,

[12] One could ask if we have not already extended the model from one to two periods. In the standard version of the CAPM there is no C_0. The agent uses all wealth on the portfolio. For our purpose it is easier to include the utility of consumption in period 0. But it is not surprising that we have derived the standard CAPM from our assumptions: For any given magnitude of $W_0 - C_0$, the agent will face the choice of an optimal portfolio. If there are no exogenous sources of income in period 1, this subproblem is exactly the one solved by the agents in the CAPM.

quantity, are uncertain. The product pq will now replace P_{j1} in (28). Assume furthermore that q is stochastically independent of the vector (p, C_1^h). This implies that $E(pq) = E(p)E(q)$ and that $\text{cov}(pq, C_1^h) = E(q)\,\text{cov}(p, C_1^h)$, so that $E(q)$ may be factored out of the square brackets in (28). Accordingly only the expected q matters.

Finally it may be in place to comment on the relation between this model and the well-known division between complete and incomplete markets under uncertainty.[13] The markets are called complete if there exists a state-contingent claim for each state, or other securities giving equally good opportunities for diversification.

Observe first that if everyone has free access to such markets, the existence of non-marketable assets has no bearing on the diversification possibilities, since one may acquire a portfolio which is perfectly negatively correlated with any non-marketable asset.

The standard CAPM represents a restriction with regard to which markets are assumed to exist. Agents in the CAPM do not have the opportunity to compose any pattern of income in different states. When the model is used to evaluate potential real investments, it is nevertheless common to assume that these will be so small in relation to the whole economy that they may be valued as if they will not affect the probability distribution of the return on the market portfolio.[14]

The model of the present paper is further removed from the assumption of complete markets, by allowing individual differences in diversification opportunities. But it sticks to the assumption that the project is small, so that it may be evaluated based on marginal considerations.

5 Conclusion

The new result in this paper is the criterion for investing in more of an asset holding which at the outset is exogenously given. It is shown that the criterion may be expressed from observable magnitudes based on certain assumptions.

Beside the specific formulae derived, one well-known main message stands out: The relevant measure of risk in an investment decision is a covariance measure. It is usually misleading to evaluate the risk of an investment in isolation, e.g., measured by the variance of the return. This measure would be correct only if a risk averse agent has that investment as her only source of uncertain income. When there are more sources of uncertainty, one should consider the contribution of each to the total uncertainty.

We have seen that the covariance measure is *not* a result of introducing the assumption that agents only care about the means and the variances of their uncertain incomes. Already in the equations (9) and (14) the risk was measured by a covariance expression. It is also not the case that agents should be able to compose an optimal portfolio for a covariance measure to be appropriate: (14) holds even if it is impossible to choose the holding of any risky asset optimally.

On the other hand, the simplifying assumptions (A1) or (A2) will be useful to find a numerical expression for the investment criterion. Alternatively one might have introduced other assumptions, e.g., on the form of the utility function.

[13]In section VIII of Constantinides (1989) there are exogenous assets which lead to markets not being complete. I prefer to distinguish the case of exogenous assets from other situations with non-complete markets, like the situation in the standard CAPM.

[14]See, e.g., the end of footnote 10 in Rubinstein (1973).

References

Brito, N.O. 1977, "Marketability restrictions and the valuation of capital assets under uncertainty," *Journal of Finance 32,* 1109–1123.

Bøhren, Ø., Ekern, S.E. 1987, "Usikkerhet i oljeprosjekter. Relevante og irrelevante risikohensyn," (Uncertainty in oil projects. Relevant and irrelevant risk considerations) *Beta 1,* 23–30.

Constantinides, G.M. 1989, "Theory of Valuation: Overview and Recent Developments," in Bhattacharya, S., Constantinides, G.M., eds., *Theory of Valuation. Frontiers of Modern Financial Theory, Volume 1,* Rowman and Littlefield, Totowa, New Jersey, 1–23.

Gravelle, H., Rees, R. 1992, *Microeconomics,* 2nd ed. Longman, London.

Huang, C.-f., Litzenberger, R.H. 1988, *Foundations for Financial Economics,* North-Holland, Amsterdam.

Lintner, J. 1965, "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets," *Review of Economics and Statistics 47,* 13–27.

Lommerud, K.E. 1984, "Velferdsøkonomisk perspektiv på olje og usikkerhet," (Welfare economic perspectives on oil and uncertainty) *Statsøkonomisk tidsskrift 98,* 41–65.

Lund, D. 1988, "Social discount rates under uncertainty — a reexamination and extension of Sandmo's 'farm' model," Memorandum No. 10/88, Dept. of Economics, University of Oslo, July 6.

Lund, D. 1990, "Social discount rates for Norwegian oil projects under uncertainty," in Bjerkholt, O., Olsen, Ø., Vislie, J., eds., *Recent Modelling Approaches in Applied Energy Economics,* Chapman and Hall, London, 171–186.

Lund, D. 1993, "Samfunnsøkonomisk vurdering av usikkerhet," (Social valuation of uncertainty) *Norsk økonomisk tidsskrift 107,* 155–178.

Mayers, D. 1972, "Nonmarketable assets and capital market equilibrium under uncertainty," in Jensen, M.C., ed., *Studies in the Theory of Capital Markets,* Praeger, New York, 223–248.

Mossin, J. 1966, "Equilibrium in a capital asset market," *Econometrica 35,* 768–783.

Ross, S.A. 1978, "Mutual fund separation in financial theory — The separating distributions," *Journal of Economic Theory 17,* 254–86.

Rubinstein, M. 1973, "A mean-variance synthesis of corporate financial theory," *Journal of Finance 28,* 167–181.

Rubinstein, M. 1976, "The valuation of uncertain income streams and the pricing of options," *Bell Journal of Economics 7,* 407–425.

Sandmo, A. 1972, "Discount Rates for Public Investment under Uncertainty," *International Economic Review 13,* 287–302.

Sharpe, W.F. 1964, "Capital asset prices: A theory of market equilibrium under conditions of risk," *Journal of Finance 19,* 425–442.

Stein, C. 1973, "Estimation of the mean of a multivariate normal distribution," *Proceedings of the Prague Symposium on Asymptotic Statistics,* September.