Asset pricing

Tries to understand the prices or values of claims to uncertain payments.

If stocks have an average real return of about 8%, then 2% may be due to interest rates and the remaining 6% is a premium for holding risk.

Price equals expected discounted payoff:

$$p_t = E\left(m_{t+1}x_{t+1}\right)$$

where p_t = asset price, x_{t+1} = asset payoff, m_{t+1} = stochastic discount factor.

Basic consumption based model

We model investors by a utility function defined over current and future values of consumption

$$U\left(c_t, c_{t+1}\right) = u\left(c_t\right) + \beta E_t\left[u\left(c_{t+1}\right)\right]$$

where the period utility function $u(.)$ is increasing and concave in consumption levels. This captures investors impatience (through $\beta < 1$) and their aversion to risk (through $u'' < 0$).

If you buy a stock today, the payoff next period is the stock price plus dividend

$$\begin{aligned} x_{t+1} &= p_{t+1} + d_{t+1} \\ x_{t+1} &= 1 + r_{t+1}^{i} \text{ (Romer with } p = 1\text{)} \\ x_{t+1} &= \sum_{i=1}^{n} p_{it+1} + y_{it+1} \text{ (Williamson)} \end{aligned}$$

Optimizing

Assume that the investor can freely buy and sell as much of the payoff x_{t+1} as we wishes, at a price p_t. How much will he buy and sell? Denote by e the original consumption level (if the investor bought none of the asset), and denote by z the amount of the asset he chooses to buy.

$$\max_{z_{t+1}} u\left(c_t\right) + \beta E_t\left[u\left(c_{t+1}\right)\right] \ s.t.$$

$$\begin{aligned} c_t &= e_t - p_t z_{t+1} \\ c_{t+1} &= e_{t+1} + z_{t+1} x_{t+1} \end{aligned}$$

(In Williamson, assets are the only income sources, i.e.

$$\begin{aligned} c_t &= z_t x_t - p_t z_{t+1} \\ c_{t+1} &= z_{t+1} x_{t+1} + p_{t+1} z_{t+2} \ ) \end{aligned}$$

Stochastic discount factor

Substituting the constraints into the objective, and setting the derivative with respect to z_{t+1} equal to zero, we obtain the first order conditions for an optimal consumption and portfolio choice

$$p_t u'(c_t) = E_t\left[\beta u'(c_{t+1})\, x_{t+1}\right]$$

or

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] = E_t\left(m_{t+1} x_{t+1}\right)$$

i.e. $m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$, where m is the stochastic discount factor $=$ intertemporal marginal rate of substitution.

Rate of return

Denote the gross rate of return on a share between time t and $t+1$ as

$$R_{t+1} = \frac{x_{t+1}}{p_t} \; (\underbrace{= \pi_{it}}_{\text{Williamson}} \; \underbrace{= 1 + r_{t+1}^{i}}_{\text{Romer}})$$

(we can think of return as payoff with price one). Then we can rewrite the pricing equation as

$$1 = E_t\left(m_{t+1} R_{t+1}\right)$$

and, using the fact that for any two variables x and y, $cov\left(x, y\right) = E\left(xy\right) - E\left(x\right) E\left(y\right)$,

$$1 = E_t\left(m_{t+1}\right) E_t\left(R_{t+1}\right) + cov\left(m_{t+1}, R_{t+1}\right)$$

Risk-free rate of return

Remember the marginal rate of substitution from last lecture

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + \bar{r} = \bar{R} = \frac{1}{E(m_{t+1})}$$

Inserting this in

$$p_t = E_t(m_{t+1}) E_t(x_{t+1}) + cov(m_{t+1}, x_{t+1})$$

we obtain

$$p_t = \frac{E_t(x_{t+1})}{\bar{R}} + cov(m_{t+1}, x_{t+1})$$

The first term is the standard discounted present value formulation, the second term is a risk adjustment.

Consumption CAPM

Applying the covariance decomposition for expected returns and using $\bar{R} = 1/E(m_{t+1})$:

$$\begin{aligned} E_t(m_{t+1})\, E_t(R_{t+1}) - 1 &= -cov(m_{t+1}, R_{t+1}) \\ E_t(R_{t+1}) - \bar{R} &= -\bar{R}\, cov(m_{t+1}, R_{t+1}) \end{aligned}$$

or

$$E_t(R_{t+1}) - \bar{R} = -\frac{cov\left[u'(c_{t+1}), R_{t+1}\right]}{E\left[u'(c_{t+1})\right]}$$

Assets whose returns covary positively with consumption make consumption more volatile and must promise investors higher expected returns to induce investors to hold them. The covariance between an asset's return and consumption is known as its *consumption beta*.

Longer horizon

It is convenient to use only the two-period valuation, thinking of a price p_t and a payoff, x_{t+1}. But there are times where we want to relate the price to the entire cash flow stream. A longer term obejctive would be

$$E_t \sum_{s=t+1}^{\infty} \beta^{s-t} u\left(c_s\right)$$

Now suppose an investor can purchase a stream $\{d_s\}$ at price p_t. As with the two-period model, his first order conditions gives ut the pricing formula directly

$$p_t = E_t \left[\sum_{s=t+1}^{\infty} \frac{\beta^{s-t} u'\left(c_s\right)}{u'\left(c_t\right)} d_s \right] = E_t \left[\sum_{s=t+1}^{\infty} m_s d_s \right]$$

Longer horizon, cont.

In other words, we can write the current share price for any asset as the expected discounted value of future dividends, where the discount factors are the intertemporal marginal rates of substitution.

$$p_t = E_t\left[\sum_{s=t+1}^{\infty} m_s d_s\right]$$

If this equation holds at time t and $t+1$ then we can derive the two-period version

$$p_t = E_t\left[m_{t+1}\left(p_{t+1} + d_{t+1}\right)\right] = E_t\left[m_{t+1} x_{t+1}\right]$$

Example: risk neutrality

Remember the Euler equation

$$p_t = E_t\left[\beta\left(u'\left(c_{t+1}\right)/u'\left(c_t\right)\right)x_{t+1}\right]$$

If there is risk neutrality, then $u'(c) = c$ and

$$p_t = \beta E_t\left[x_{t+1}\right] = \beta E_t\left[p_{t+1} + d_{t+1}\right]$$

The current price is equal to the discounted value of expected price plus the dividend for the next period. If $d = 0$ and β close to 1:

$$\begin{aligned} E\left(p_{t+1}\right) &= p_t \\ p_{t+1} &= p_t + \varepsilon_{t+1} \end{aligned}$$

Prices follow a random walk, and the rate of return is unpredictable using current information ($E\left(R_{t+1}\right) = 1/\beta = 1$).

Example: constant risk aversion

Consider preferences of the form

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

With this assumption, marginal utility is $u' = c^{-\gamma}$, and the Euler equation becomes

$$\begin{aligned} p_t &= E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} x_{t+1}\right] \\ 1 &= E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} R_{t+1}\right] \end{aligned}$$

The growth rate of consumption

Denote the growth rate of consumption

$$g_c = \frac{c_{t+1}}{c_t} - 1$$

drop the time subscripts, and use that $R = (1 + r)$ and $\beta = 1/(1 + \rho)$:

$$1 + \rho = E\left[(1 + g_c)^{-\gamma}(1 + r)\right]$$

We then take a second-order Taylor approximation on the right hand side around $r = g = 0$:

$$\begin{aligned} 1 + \rho &\approx E\left[1 + r - \gamma g_c - \gamma g_c r + \frac{1}{2}\gamma(\gamma + 1) g_c^2\right] \\ \rho &\approx E(r) - \gamma E(g_c) - \gamma\left[E(r) E(g_c) + cov(r, g_c)\right] \\ &\quad + \frac{1}{2}\gamma(\gamma + 1)\left\{\left[E(g_c)\right]^2 + var(g_c)\right\} \end{aligned}$$

Equity-premium puzzle

Using that $E(r)\,E(g_c)$ and $[E(g_c)]^2$ are small and solving for $E(r)$

$$E(r) \approx \rho + \gamma E(g_c) + \gamma cov(r, g_c) - \frac{1}{2}\gamma(\gamma+1)\,var(g_c)$$

In the risk-free case the reduces to

$$\bar{r} \approx \rho + \gamma E(g_c) - \frac{1}{2}\gamma(\gamma+1)\,var(g_c)$$

Finally, subtracting the risk-free rate from the risky rate yields

$$E(r) - \bar{r} \approx \gamma cov(r, g_c)$$

Observed asset returns and growth rates of consumption results in estimates of γ that imply extraordinary and unrealistic levels of risk aversion: *Equity-premium puzzle.*

Equity-premium puzzle: explanations

Estimates of γ based on observed asset returns and growth rates of consumption are somewhere between $\gamma = 25$ (Mehra and Prescott) and $\gamma = 91$ (Mankiw and Zeldes), while in calibrations of life-cycle models it is common to assume γ somewhere around 4.

Possible explanations:

Transaction costs: Trading in stocks involve entry and exit costs. Also, there may be trading constraints.

Habit persistence: consumption fluctuations can have a very large effect of the marginal rate of substitution even if γ is not high.

"Simply a historical anomaly of the post-Depression period."

Summing up the basic equations

$$
\begin{aligned}
p_t &= E_t\left[\sum_{s=t+1}^{\infty} \frac{\beta^{s-t} u'(c_s)}{u'(c_t)} d_s\right] \\
m_{t+1} &= \beta \frac{u'(c_{t+1})}{u'(c_t)} \\
p &= E(mx) \\
E(R) &= \bar{R} - \bar{R}\, cov(m, R) \\
E(r) - \bar{r} &\approx \gamma cov(r, g_c)
\end{aligned}
$$