ECON4310 Fall 2011 Seminar 1, solution sketch

Week 41

1 Ramsey's growth model

1. $k_t = K_t/A_tL_T$ measures the amount of capital stock per unit of effective labor at time t, c_t measures the level of consumption per unit of effective labor at time t.

   Social planner's problem:

$$\max_{c_t} U = \max \sum_{t=0}^{\infty} \beta^t u(\bar{c}_t) = \max \sum_{t=0}^{\infty} \beta \bar{c}_t^{1-\theta}/(1-\theta) \quad (1)$$

$$= \max \sum_{t=0}^{\infty} \beta^t \frac{(A_t c_t)^{1-\theta}}{1-\theta} \quad (2)$$

subject to budget constraints:

$$K_{t+1} = K_t - \delta K_t + Y_t - C_t$$

divide both sides by A_tL_t, we have

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}}(1+n)(1+g) = (1-\delta)\frac{K_t}{A_tL_t} + \frac{Y_t}{A_tL_t} - \frac{C_t}{A_tL_t}$$

which is

$$(1+n)(1+g)k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$$

2. value function:

$$V(k_t, A_t) = \max \sum_{s=t}^{\infty} \beta^t \frac{(A_s c_s)^{1-\theta}}{1-\theta}$$

$$\text{subject to.} c_t = -(1+n)(1+g)k_{t+1} + (1-\delta)k_t + f(k_t),$$

   Bellman equation:

$$V(k_t, A_t) = \max_{k_{t+1}}(u((1-\delta)k_t+f(k_t)-(1+n)(1+g)k_{t+1})A_t)+\beta V(k_{t+1}, A_{t+1}))$$

First order condition, at the optimal solution, we have

$$A_t u'(c_t^* A_t)(1+n)(1+g) = \beta \frac{\partial V'(k_{t+1}^*, A_{t+1})}{\partial k_{t+1}} \tag{3}$$

using envelop theorem; we also have

$$\frac{\partial V'(k_{t+1}^*, A_{t+1})}{\partial k_{t+1}} = A_{t+1}u'(c_{t+1}^* A_{t+1})(1-\delta+f'(k_{t+1}^*))$$

So we have

$$\frac{A_t u'(c_t^* A_t)}{A_{t+1}u'(c_{t+1}^* A_{t+1})} = \frac{(1-\delta+f'(k_{t+1}^*))}{(1+n)(1+g)}$$

note that $u'(c_t^* A_t) = A_t^{-\theta}(c_t^*)^{-\theta}$, $u'(c_{t+1}^* A_{t+1}) = A_t^{-\theta}(1+g)^{-\theta}(c_{t+1}^*)^{-\theta}$, we have

3. Balanced growth path: each of variable of the model is growing at a constant rate. In other words, along the balanced growth path c_t, k_t are constants. Then we have

   $$1 = \beta \frac{1+f'(k)-\delta}{(1+g)^\theta(1+n)} = \frac{1}{(1+g)^\theta} \frac{1+f'(k)-\delta}{(1+\rho)(1+n)}$$

   So $f'(k) = (1+g)^\theta(1+n)(1+\rho)+\delta-1$. so the steady state solution depends on only the growth rate g, n the discount rate β and the depreciation rate δ.

   since $f''(k)$ is negative, so a increase of the parameter θ will lead to a decrease on the steady state value k.

4. reduced population growth leads to a higher level of capitial accumlation per unit of effective labor as well as higher level of consumption per unit of effective labor. (showed in the phase plane diagram)

   note that real wage is defined as

   $$W(t) = \frac{\partial F}{\partial L} = A(f(k(t)) - kf'(k(t)))$$

   so we have

   $$\frac{\partial W}{\partial k} = -Akf''(k) > 0$$

   which means that higher capitial accumlation leads to higher real wage. So in our model, reduced population growth leads to higher real wage.

5. If for the starting points near the equilibirum, the solution goes to the equilibirum then the equilirium is stable, otherwise, it is not stable.

   The steady state in Ramsay Model is a Saddle point. (Showed on the phase plane diagram)

   For any given value of $k(0) > 0$, there exists one and only one value $c(0)$ such that the solution converge to the stable state. (Showed on the phase plane)

6. See the phase diagram showed in seminar.

2 The Golden Rule of Accumulation and Dynamic Inefficiency

1. It is discussed in Romer p62-63.

2. Dynamic Inefficiency: the model equilibirum is not pareto efficient, despite of competive and no externatility. It arises when the economy accumulate too much capitial ($>$ golden-rule level). It can never occur in the Ramsey model discussed above. It can happen in Diamond Model. See the discussion in the book.