# CRRA-utility

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The Constant Relative Risk Aversion (CRRA) utility function is

$$u(c) = \begin{cases} \frac{1}{1-\theta} c^{1-\theta} & \text{if } \theta > 0, \theta \neq 1 \\ \ln c & \text{if } \theta = 1 \end{cases}$$

The parameter $\theta$ measures the degree of relative risk aversion that is implicit in the utility function. Below we will focus on other properties of the function.

Suppose we have two goods and that

$$U = u(c_1) + u(c_2)$$

Since the first derivative of the CRRA utility function is

$$u'(c) = c^{-\theta},$$

the marginal rate of substitution is

$$\frac{u'(c_1)}{u'(c_2)} = \frac{c_1^{-\theta}}{c_2^{-\theta}} = \left(\frac{c_2}{c_1}\right)^{\theta}$$

or, solving for $c_2/c_1$.:

$$\frac{c_2}{c_1} = \left(\frac{u'(c_1)}{u'(c_2)}\right)^{1/\theta}$$

Hence, $1/\theta$ is the elasticity of the ratio of the consumed quantities of the two goods with respect to the marginal rate of substitution. By definition $\sigma = 1/\theta$ is then the elasticity of substitution, which is constant for the CRRA utility function. $\sigma$ is a measure of the strength of the substitution effect that a change in relative prices induces. This is evident when we remember that the first order condition for the consumer makes the marginal rate of substitution equal to the price ratio. Hence,

$$\frac{c_2}{c_1} = \left(\frac{p_1}{p_2}\right)^{\sigma}$$

This also reveals another important property that follows when total utility is a sum of CRRA functions. The ratios between the consumption of different goods depend only on relative prices, not on the income level. This means that if there is an increase in real income, the consumption of all goods will go up in the same proportion as income. In other words, all income elasticities are equal to one.

Figure 1 sketches what the CRRA-function looks like depending on whether $\theta$ is above, below or equal to 1.

Marginal utility is always positive and diminishing:

$$u'(c) = c^{-\theta} > 0, \qquad u''(c) = \theta c^{-\theta-1} < 0$$

You can check that these expressions are valid also for the special case $\theta = 1$. From these we see that the marginal utility goes to infinity when $c$ goes to zero. At the opposite end the marginal utility goes to zero when $c$ goes to infinity. When $\theta > 1$, the utility level is bounded above, but not below. When $\theta < 1$, it is the opposite; the utility level is bounded below, but not above. In the logarithmic case the utility level is neither bounded above of below. The utility level itself is of course arbitrary and and without consequences for behavior.

## CRRA in the Ramsey model

In the context of the Ramsey model a low $\sigma$ means a strong preference for avoiding inequality between generations in excess of what follows from the discounting in the utility function. That all income elasticities are equal to one makes it possible to have balanced growth paths also when there is productivity growth.

The first order condition (Euler equation) in the Ramsey model without natural growth was:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta[1 + f'(k_{t+1})]$$

With CRRA utility this becomes

$$\left(\frac{c_{t+1}}{c_t}\right)^\theta = \beta[1 + f'(k_{t+1})]$$

which means that the consumption growth rate is

$$\frac{c_{t+1}}{c_t} = \left[\beta(1 + f'(k_{t+1}))\right]^{1/\theta} = \left(\frac{1 + f'(k_{t+1})}{1 + \rho}\right)^\sigma$$

A high $\sigma$ means the difference between the marginal productivity of capital and the subjective discount rate has a strong effect on the consumption growth rate.

If there is exponential productivity growth forever, it is possible to make consumption go towards infinity as time goes to infinity. When $\theta > 1$ this raises the prospect of infinite total utility.

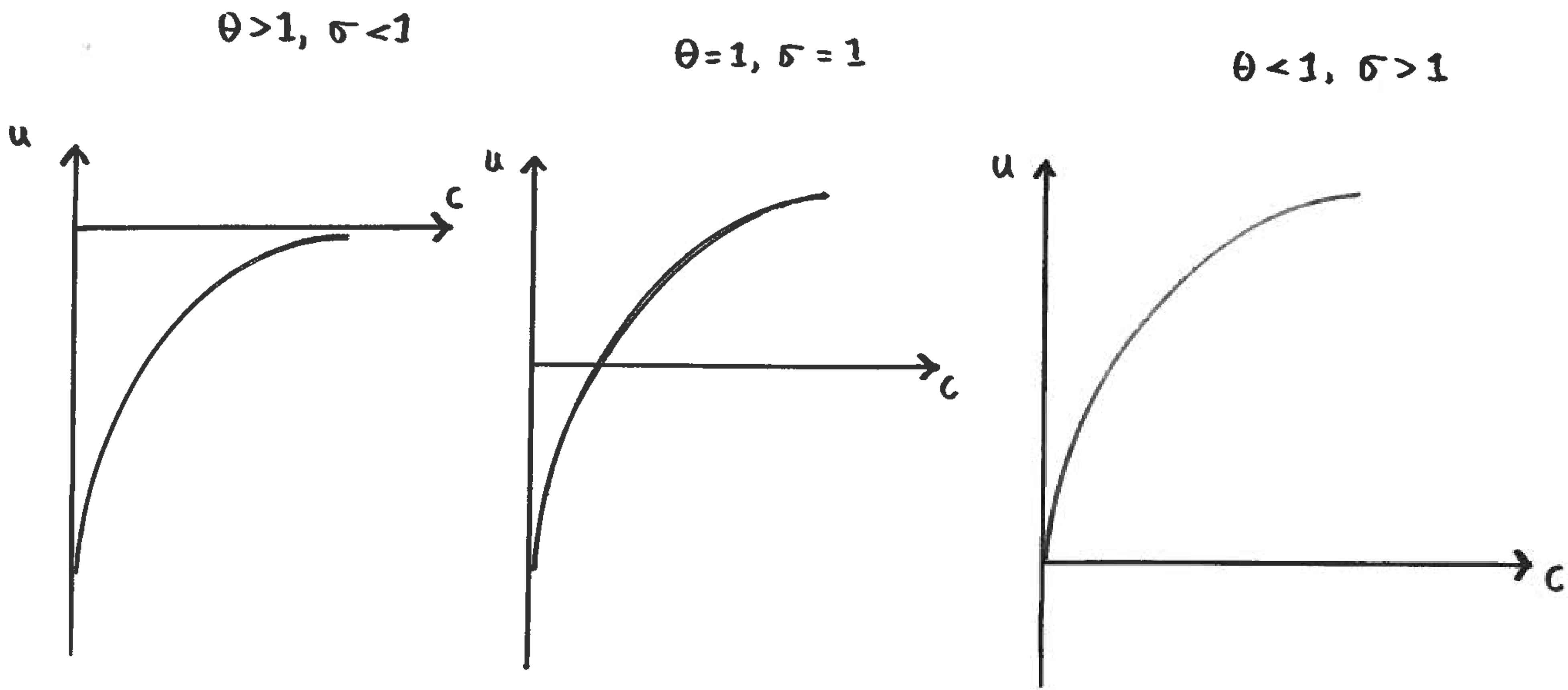


Figure 1