15 Asset pricing

Asset pricing theory tries to understand the prices or values of claims to uncertain payments. To value an asset we have to account for the *delay* and for the *risk* of its payments.

Asset price equals expected discounted payoff.

Consumption CAPM: Asset premiums are proportional to the covariance between an asset's return and consumption.

Equity puzzle: If stocks have an average real return of about 8%, then 2% may be due to interest rates and the remaining 6% is a premium for holding risk. Are investors really that risk averse?

Prices, payoff and notation

Stocks/assets

Price: p_t ($= 1$ in Romer)

Payoff: $x_{t+1} = p_{t+1} + y_{t+1}$, where y_{t+1} is dividend (usually denoted d_t or d_{t+1}). Williamson assumes a sum of assets; $x_{t+1} = \sum_{i=1}^{n} (p_{it+1} + y_{it+1})$

Return/profit/gains

$$R_{t+1} = \frac{x_{t+1}}{p_t}$$

where $R = 1 + r$ is gross return, whereas $r = R - 1$ is net return. If $p_t = 1$ then $R_{t+1} = x_{t+1}$.

Returns re-written

$$
\begin{aligned}
R_{t+1} &= \frac{x_{t+1}}{p_t} \\
1 + r_{t+1} &= \frac{p_{t+1} + y_{t+1}}{p_t} \\
r_{t+1} &= \frac{p_{t+1} - p_t + y_{t+1}}{p_t} \\
r_{t+1} p_t &= \underbrace{p_{t+1} - p_t}_{\text{capital gain}} + \overbrace{y_{t+1}}^{\text{dividend}}
\end{aligned}
$$

Basic consumption based model

We model investors by a utility function defined over current and future values of consumption

$$U(C_t, C_{t+1}) = u(C_t) + \beta E_t [u(C_{t+1})]$$

where the period utility function $u(.)$ is increasing and concave in consumption levels. This captures investors *impatience* (through $\beta < 1$) and their *aversion to risk* (through $u'' < 0$). Assume that the investor can freely buy and sell as much of the asset as he wishes, at a price p_t.

Denote by e the original consumption level (if the investor bought none of the asset), and denote by z the amount of the asset he chooses to buy.

$$\max_{z} u(C_t) + \beta E_t [u(C_{t+1})] \ \ s.t.$$

$$\begin{aligned} C_t &= e_t - p_t z_{t+1} \\ C_{t+1} &= e_{t+1} + z_{t+1} x_{t+1} \end{aligned}$$

Note: In Williamson, assets are the only income source, i.e.

$$\begin{aligned} C_t &= z_t x_t - p_t z_{t+1} \\ C_{t+1} &= z_{t+1} x_{t+1} + p_{t+1} z_{t+2} \end{aligned}$$

First order condition

Substituting the constraints into the objective, and setting the derivative with respect to z_{t+1} equal to zero, we obtain the first order conditions for an optimal consumption and portfolio choice

$$p_t u'(C_t) = E_t\left[\beta u'(C_{t+1}) x_{t+1}\right]$$

or

$$p_t = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} x_{t+1}\right]$$

Price equals discounted payoff.

Stochastic discount factor

Define m, the stochastic discount factor (equal to the intertemporal marginal rate of substitution):

$$m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

and re-write the first order condition

$$p_t = E_t [m_{t+1} x_{t+1}]$$

This is the basic asset pricing formula.

Asset pricing

Remember that the gross rate of return is given by

$$R_{t+1} = \frac{x_{t+1}}{p_t}$$

We can rewrite the asset pricing equation as

$$1 = E_t\left(m_{t+1}R_{t+1}\right)$$

This is the most important special case of the basic formula $p = E(mx)$. The expectation of a product is the product of the expectations plus their covariance:

$$1 = E_t\left(m_{t+1}\right)E_t\left(R_{t+1}\right) + Cov\left(m_{t+1}, R_{t+1}\right)$$

Risk-free rate of return

If we assume that there exists a risk-free asset, $\bar{R}$, this asset will have zero covariance with the stochastic discount factor:

$$\bar{R} = \frac{1}{E\left(m_{t+1}\right)}$$

Inserting this in

$$p_t = E_t\left(m_{t+1}\right) E_t\left(x_{t+1}\right) + Cov\left(m_{t+1}, x_{t+1}\right)$$

we obtain

$$p_t = \frac{E_t\left(x_{t+1}\right)}{\bar{R}} + Cov\left(m_{t+1}, x_{t+1}\right)$$

The first term is the standard discounted present value formulation, the second term is a risk adjustment.

Consumption CAPM

Applying the covariance decomposition for expected returns and using $\bar{R} = 1/E(m_{t+1})$:

$$\begin{aligned} E_t(m_{t+1})\,E_t(R_{t+1}) - 1 &= -Cov(m_{t+1}, R_{t+1}) \\ E_t(R_{t+1}) - \bar{R} &= -\bar{R}\,Cov(m_{t+1}, R_{t+1}) \end{aligned}$$

or

$$E_t(R_{t+1}) - \bar{R} = -\frac{Cov\left[u'(C_{t+1}), R_{t+1}\right]}{E\left[u'(C_{t+1})\right]}$$

Assets whose returns co-vary positively with consumption make consumption more volatile and must promise investors higher expected returns to induce investors to hold them. The covariance between an asset's return and consumption is known as its *consumption beta*.

Longer horizon

It is convenient to use only the two-period valuation, thinking of a price p_t and a payoff, x_{t+1}. But there are times where we want to relate the price to the entire cash flow stream. A longer term obejctive would be

$$E_t \sum_{s=t+1}^{\infty} \beta^{s-t} u\left(C_s\right)$$

Now suppose an investor can purchase a stream $\{y_s\}$ at price p_t. As with the two-period model, his first order conditions gives ut the pricing formula directly

$$p_t = E_t\left[\sum_{s=t+1}^{\infty} \frac{\beta^{s-t} u'\left(C_s\right)}{u'\left(C_t\right)} y_s\right] = E_t\left[\sum_{s=t+1}^{\infty} m_s y_s\right]$$

Longer horizon, cont.

In other words, we can write the current share price for any asset as the expected discounted value of future dividends, where the discount factors are the intertemporal marginal rates of substitution.

$$p_t = E_t \left[\sum_{s=t+1}^{\infty} m_s y_s \right]$$

If this equation holds at time t and $t+1$ then we can derive the two-period version

$$p_t = E_t \left[m_{t+1} \left(p_{t+1} + y_{t+1} \right) \right] = E_t \left[m_{t+1} x_{t+1} \right]$$

Example: Risk neutrality

Remember the Euler equation

$$p_t = E_t \left[\beta \left(u' \left(C_{t+1} \right) / u' \left(C_t \right) \right) x_{t+1} \right]$$

If there is risk neutrality, then $u' \left(C \right) = C$ and

$$p_t = \beta E_t \left[x_{t+1} \right] = \beta E_t \left[p_{t+1} + y_{t+1} \right]$$

The current price is equal to the discounted value of expected price plus the dividend for the next period. If $y = 0$ and β close to 1:

$$\begin{aligned} E \left(p_{t+1} \right) &= p_t \\ p_{t+1} &= p_t + \varepsilon_{t+1} \end{aligned}$$

Prices follow a random walk, and the rate of return is unpredictable using current information ($E \left(R_{t+1} \right) = 1/\beta = 1$).

Example: Constant risk aversion

Consider preferences of the form

$$u(C) = \frac{C^{1-\gamma} - 1}{1-\gamma}$$

where γ is the relative risk aversion parameter. With this assumption, marginal utility is $u' = C^{-\gamma}$, and the Euler equation becomes

$$p_t = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} x_{t+1} \right]$$

or

$$1 = E_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} \right]$$

The growth rate of consumption

Denote the growth rate of consumption $g_C = C_{t+1}/C_t - 1$, drop the time subscripts, and use that $R = (1+r)$ and $\beta = 1/(1+\rho)$:

$$1 + \rho = E\left[(1+g_C)^{-\gamma}(1+r)\right]$$

We then take a second-order Taylor approximation on the right hand side around $r = g = 0$:

$$\begin{aligned} 1+\rho &\approx E\left[1 + r - \gamma g_C - \gamma g_C r + \frac{1}{2}\gamma(\gamma+1) g_C^2\right] \\ \rho &\approx E(r) - \gamma E(g_C) - \gamma\left[E(r)E(g_C) + Cov(r, g_C)\right] \\ &\quad + \frac{1}{2}\gamma(\gamma+1)\left\{[E(g_C)]^2 + var(g_C)\right\} \end{aligned}$$

Interest rates and consumption

Using that $E(r)\,E(g_C)$ and $[E(g_C)]^2$ are small and solving for $E(r)$

$$E(r) \approx \rho + \gamma E(g_C) + \gamma Cov(r, g_C) - \frac{1}{2}\gamma(\gamma+1)\,var(g_C)$$

In the risk-free case the reduces to

$$\bar{r} \approx \rho + \gamma E(g_C) - \frac{1}{2}\gamma(\gamma+1)\,var(g_C)$$

Real interest rates are high when people are impatient (ρ), when expected consumption growth is high (intertemporal substitution), or when risk is low (precautionary saving).

Equity-premium puzzle

Finally, subtracting the risk-free rate from the risky rate yields

$$E(r) - \bar{r} \approx \gamma Cov(r, g_C)$$

Observed asset returns and growth rates of consumption results in estimates of γ that imply extraordinary and unrealistic levels of risk aversion: Early estimates for the US were somewhere between $\gamma = 25$ (Mehra and Prescott) and $\gamma = 91$ (Mankiw and Zeldes). According to Mehra and Prescott a relative risk aversion parameter above 10 is unreasonable. In calibrations of life-cycle models it is common to assume γ to be maximum 4.

This is called the *equity-premium puzzle.*

The equity premium in Norway

Table 9: The Real Risk Premium over Bills for Different Periods.

Real risk premium							
Years	Mean	Std.dev	Skew	Kurt	Min	Max	$\rho\,(ep, \Delta c)$
1900-2008	5.90	26.30	2.33	11.54	-48.78	163.06	16.32
1970-2008	11.36	40.07	1.54	3.94	-48.78	163.06	19.32
1900-1970	2.76	12.95	-0.28	1.54	-40.64	36.46	23.47
1947-2008	8.59	32.53	2.01	7.42	-48.78	163.06	16.40
1900-1940	1.73	14.14	-0.26	1.88	-40.64	36.46	40.01
1914-1940	1.16	16.89	-0.19	0.65	-40.64	36.46	40.95
1947-1970	4.46	10.83	-0.15	0.16	-20.44	26.38	9.49

… and the implied risk aversion

Table 10: The Equity Premium Puzzle

Log equity and log consumption innovation							
Years	$\mu(ep)$	$\sigma(ep)$	$\sigma(\Delta c)$	$\rho(ep, \Delta c)$	Sharpe	RRA(1)	RRA(2)
1900-2008	3.12	22.34	4.14	16.32	0.25	37.21	6.07
1970-2008	5.36	32.96	2.00	19.32	0.33	84.62	16.35
1900-1970	1.90	13.27	4.89	23.47	0.21	18.27	4.29
1947-2008	4.46	26.84	1.98	16.40	0.30	92.59	15.18
1900-1940	0.69	14.77	5.57	40.01	0.12	5.41	2.16
1914-1940	0-.35	17.71	6.80	40.95	0.07	2.47	1.01
1947-1970	2.97	10.55	1.94	9.49	0.33	181.81	17.25

Possible explanations

Transaction costs: Trading in stocks involve entry and exit costs. Also, there may be trading constraints and participation costs.

Habit persistence: consumption fluctuations can have a very large effect of the marginal rate of substitution even if γ is not high.

Limited participation: lack of financial sophistication, ignorance, trust or over-investment in housing.

"Simply a historical anomaly of the post-Depression period."

Attanasio and Paiella (2011), *Journal of Applied Econometrics*, assume transaction cost and limited participation and get $\gamma = 1.7$ for the US.

Summing up the basic equations

stochastic discount factor : $m_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$

asset pricing : $p = E(mx)$

expected return : $E(R) = \bar{R} - \bar{R}\, Cov(m, R)$

equity premium : $E(r) - \bar{r} \approx \gamma Cov(r, g_C)$

Life-cycle model under uncertainty

In each period consumption is chosen so as to maximize

$$E_t\left[\sum_{s=t+1}^{T-t}\beta^{s-t}u\left(C_s\right)\right]$$

given

$$A_{t+1}=\left(1+r\right)\left(A_t+\tilde{Y}_t-C_t\right)$$

where $\tilde{Y}$ is stochastic income (the source of uncertainty). This yields the stochastic Euler equation

$$u'\left(C_t\right)=\beta\left(1+r\right)E_t\left[u'\left(C_{t+1}\right)\right]$$