ECON4310 Answers to Exercise 2

Due 6/9 2010

1. (a) Since you are asked to discuss both the steady state and the balanced growth path, you can as well start with a graph like the one on slide 10 for Lecture 1. The determination of the steady state is illustrated by the point k_* in figure 1. A reduced savings rate means that for every level of k_t we get a reduced level (shift down) in k_{t+1}. The new steady state is k'_*. If the shift happens in period t, then $k_t = k_*$. The red curve shows how k_t gradually declines towards k'_*.

   $r = f'(k)$ and since $f'' < 0$, r increases as k goes down. Hence, r increases in a step-wise fashion period by period.

   (b) The steady state is characterized by

$$sf(k_*) - \gamma k_* = 0$$

   Implicit differentiation of this equation yields

$$\frac{dk_*}{ds} = -\frac{f(k_*)}{sf'(k_*) - \gamma} > 0$$

   The denominator can be signed in the following way: The curve depicted in figure 1 is

$$k_{t+1} = [sf(k_t) + k_t]/(1+\gamma)$$

   Its slope is $[sf'(k_t) + 1]/(1+\gamma)$. At k_* this cuts the 45-degree line from above. This means that the slope there is less than one. Hence, $[sf'(k_*) + 1]/(1+\gamma) < 1$ which is the same as $sf'(k_*) < \gamma$).

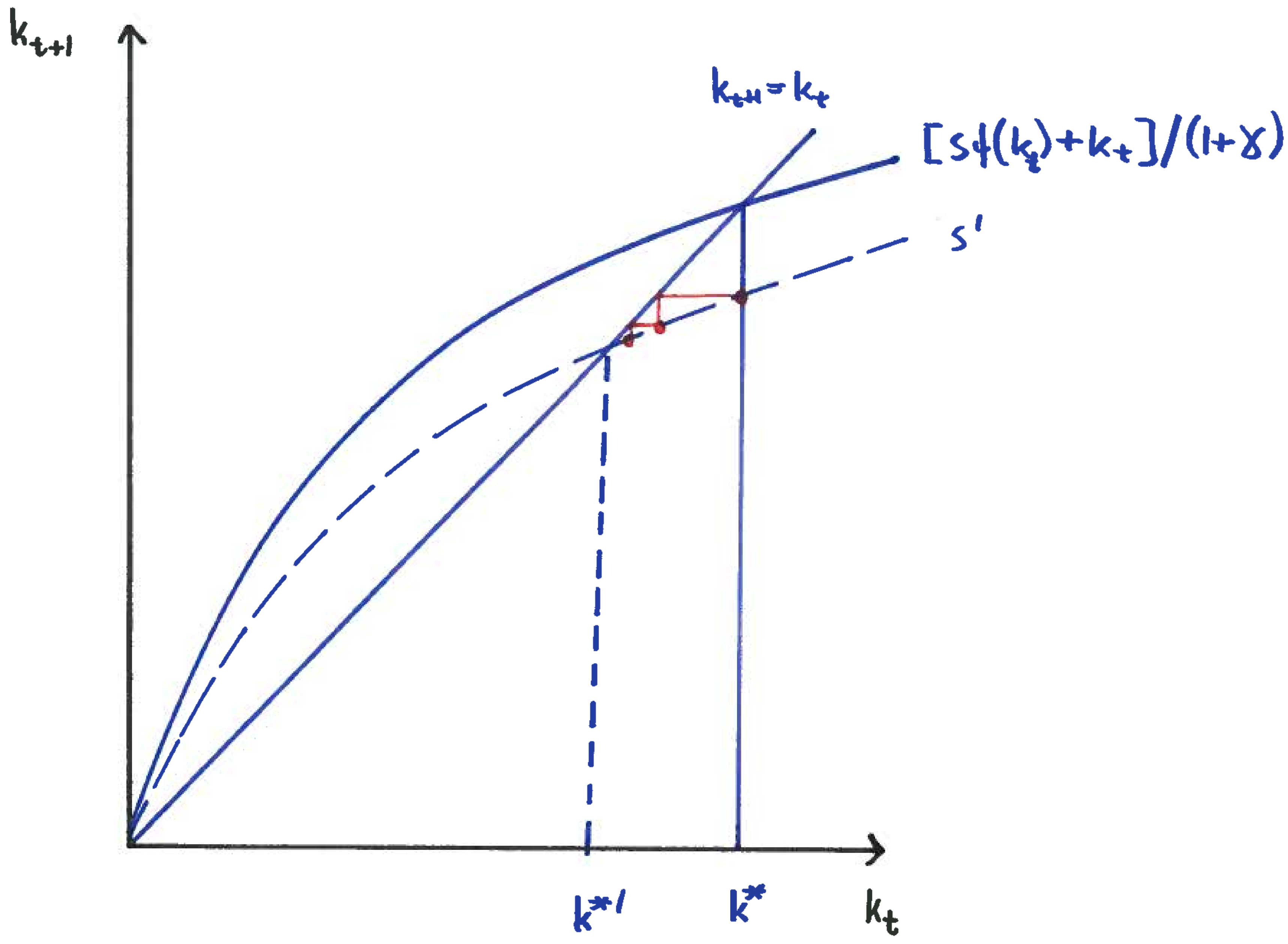
2. (a) The steady state, k_*, is determined by the condition that if $k_t = k_*$, then $k_{t+1} = k_*$. Inserting k_* for k_t and k_{t+1} in the accumulation equation yields the steady-state condition

$$(\gamma + \delta)k_* = sf(k_*)$$

   A graph like the one in slide 9 in lecture 1 may be used to illustrate.

   (b) This is the same as asking whether the two curves in the graph always intersect for a strictly positive value of k. By assumption (the Inada conditions) $sf'(0) = \infty > \gamma + \delta$ and $sf'(\infty) = 0 < \gamma + \delta$ provided that $\gamma + \delta > 0$. Hence, in this case $sf(k) > (\gamma + \delta)$ when k is close to zero while $sf(k) < (\gamma + \delta)k$ for sufficiently high ks. However, if $\gamma + \delta < 0$, the two curves can only intersect at zero, and no steady state with $k > 0$ exists. (Is this case likely in practice?)

(c) Use the graph to show that if the economy starts away from the steady state, it moves closer to the steady state every period.


k_{t+1}
$k_{t+1}=k_t$
$[sf(k_t)+k_t]/(1+\gamma)$
s'
$k^{*\prime}$
k^*
k_t

ECON4310 Answers to Exercise 3

Due 19/9 2010

1 Ramsey model with log utility

(a)
$$\frac{c_{t+1}}{c_t} = \beta[1 + \alpha(k_{t+1})^{\alpha-1}] \quad t = 0, 1, 2, \ldots$$

(Note that the production function is $f(k) = k^{\alpha}$. Hence,$f'(k) = \alpha k^{\alpha-1}$).

(b) Consumption will be growing if, and only if, $\beta[1 + f'(k_{t+1})] = \beta[1 + (k_{t+1})^{\alpha-1}] > 1$. Equivalently, consumption will be growing if the the marginal productivity of capital $f'()$ is greater than the subjective discount rate ρ (where $\beta = 1/(1+\rho)$).

(c) The steady state (k_*, c_*) is defined as the values of k_t and c_t that make k and c stay constant. From the first-order condition and the constraint (2) $k_{t+1} = k_t = k_*$ and $c_{t+1} = c_t = c_*$ requires

$$\beta[1 + \alpha k_*^{\alpha-1}] = 1, \quad \text{or} \qquad \alpha k_*^{\alpha-1} = \rho \tag{3}$$

$$c_* = k_*^{\alpha} \tag{4}$$

The explanation for $f'(k)$ ending up strictly positive is impatience (subjective discounting, $\rho > 0$) .

(d) Saving is zero since net investment is zero.

(e) Note first that with $\beta = 0.96$ $\rho = (1-\beta)/\beta = 0.042$ From (1)

$$k_* = \left(\frac{\rho}{\alpha}\right)^{-1/(1-\alpha)} = \left(\frac{0.042}{0.3}\right)^{-1/(1-0.3)} = 16.6$$

From (2) we then find $c_* = 2.3$.

With $\beta = 0.98$ we find $k_* = 45.2, c_* = 3.13$.

2 Optimal growth with equal weight for all generations

(a) Focus on the terms in (3) that contain k_{t+1}:

$$U_0 = \ldots + \beta^t u(c_t A_t) + \beta^{t+1} u(c_{t+1} A_{t+1}) + \ldots$$

Take the derivative with respect to k_{t+1} and set it equal to zero to get the first order conditions:

$$\frac{\partial V}{\partial k_{t+1}} = \beta^t u'(c_t A_t) A_t \frac{dc_t}{dk_{t+1}} + \beta^{t+1} u'(c_{t+1} A_{t+1}) A_{t+1} \frac{dc_{t+1}}{dk_{t+1}} = 0$$

Replace dc_t/dk_{t+1} and dc_{t+1}/dk_{t+1} with the expressions you get from (4):

$$-\beta^t u'(c_t A_t)(1+n)(1+g) A_t + \beta^{t+1} u'(c_{t+1} A_{t+1}) A_{t+1} (1 + f'(k_{t+1}))$$

Divide through by $\beta^t A_{t+1}$ and change sides:

$$u'(c_t A_t)(1+n) = \beta u'(c_{t+1} A_{t+1})(1 + f'(k_{t+1})) = 0$$

Replace u' with the expressions we get from differentiating the CRRA utilty functions:

$$\frac{c_{t+1}}{c_t} = \left[\frac{1 + f'(k_{t+1})}{(1+\rho)(1+n)}\right]^\sigma \frac{1}{1+g}, \qquad t = 1, 2, \ldots$$

(b) Along a balanced growth path $c_{t+1} = c_t$ and $k_{t+1} = k_t = k_*$ The first order condition above then boils down to

$$\left[\frac{1 + f'(k_*)}{(1+\rho)(1+n)}\right]^\sigma \frac{1}{1+g} = 1$$

A log-linear approximation is

$$(i) \qquad f'(k_*) \approx \rho + n + \frac{1}{\sigma} g$$

This can be compared to what we get with strictly utilitarian preferences

$$(ii) \qquad f'(k_*) \approx \rho + \frac{1}{\sigma} g$$

and the golden rule

$$(iii) \qquad f'(k_*) \approx n + g$$

With $\sigma < 1$ and $\rho > n$ (which ensures that the maximization is meaningful in all cases) the required return on capital is highest in case (i), lowest in case (iii). The level of k_* is then highest in case capital stock in case (iii). If we compare (i) and (iii) the reasons for the higher return requirement and lower savings in the first case are two: *Impatience* ($\rho > 0$) and a *strong preference for equality* $\sigma < 1$). Both reduce savings, the latter because $g > 0$ is supposed to make later generations better off anyway. In the intermediate case the higher weight on the larger future generations dampens the effect of discounting.

(c) a) zero, b) g c) $g + n$

(d) Phase diagram as shown in the lectures, but with stationary point to the left. Starting point to the right of the stationary point. Curve for $k_{t+1} = k_t$ is given by

$$c_t = f(k_t) - (n + g + ng)k_t$$

The position of the vertical curve for $c_{t+1} = c_t$ is given in the answer to (b). Explain briefly how the starting point is pinned down by drawing one unsustainable and one inefficient path starting from different consumption levels but the same k.

(e) Curve for c constant shifts to the left, since the required return on capital increases (see (b)). Curve for k constant shifts down. New stationary point is down and to the left of the old.

ECON4310 Answers Exercise 4

Due 26/9 2011

CRRA utility and consumption choice over two periods

1. $u'(c) = c^{-\theta} > 0$ and $u''(c) = -\theta c^{-(1-\theta)} < 0$.

2. $u'(c) \to \infty$ when $c \to 0$, $u'(c) \to 0$ when $c \to \infty$.

3. A positive ρ means a preference for consumption in period 1 over consumption in period 2. If the consumer is asked to distribute a given quantity of goods between the two periods, he will prefer to allocate more than one half to period two, and more the higher is ρ. σ describes the willingness of the consumer to give up consumption in one period for consumption in the other period. A low substitution elasticity means that the compensation a consumer demands in terms of consumption in say period 1 for giving up consumption in say period 2 increases rapidly with the initial ratio between consumption in periods 1 and 2. Graphically a low σ means that the indifference curves are strongly curved. A low σ means an aversion to large differences in consumption between the two periods. [There are, of course, many ways to formulate the answer to this question].

4. First-order conditions can be found either by Lagrange's method or by using the budget equation to replace c_1 or c_2 in the utility function. It boils down to

$$\frac{(1+\rho)c_1^{-\theta}}{c_2^{-\theta}} = 1 + r \Leftrightarrow \frac{c_2}{c_1} = \left(\frac{1+r}{1+\rho}\right)^{\sigma}$$

The condition can be expressed in many different ways, but these two version are often chosen because of their interpretation, the first as saying that the marginal rate of substitution should be equal to the price ratio, the second saying that the growth rate of consumption depends on the relation between the market rate of interest and the consumer's degree of impatience (subjective discount rate).

Her is an example of a detailed derivation: If we solve the budget constraint for c_2, we get:

$$c_2 = (1 + r)(w - c_1)$$

Insert this in the utility function:

$$V = u(c_1) + \frac{1}{1+\rho} u((1+r)(w-c_1))$$

By differentiating with respect to c_1 we then get the first order condition

$$\frac{dV}{dc_1} = u'(c_1) - \frac{1}{1+\rho} u'((1+r)(w-c_1))(1+r) = 0$$

or, after moving the last term to the other side

$$u'(c_1) = \frac{1}{1+\rho} u'(c_2)(1+r)$$

Replace $u'(\cdot)$ by the derivative of the CRRA-function:

$$c_1^{-\theta} = \frac{1+r}{1+\rho} c_2^{-\theta}$$

Multiply by c_2^{θ} on both sides and you get

$$\left(\frac{c_2}{c_1}\right)^{\theta} = \frac{1+r}{1+\rho}$$

Solving this for c_2/c_1 yields

$$\frac{c_2}{c_1} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = \left(\frac{1+r}{1+\rho}\right)^{\sigma}$$

5. Use the first-order condition to replace c_2 in the budget constraint:

$$c_1 + \frac{1}{1+r}\left(\frac{1+r}{1+\rho}\right)^{\sigma} c_1 = w$$

Solve this linear equation for c_1:

$$c_1 = \frac{(1+\rho)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$

Insert the solution for c_1 in the first order condition to find c_2.

$$c_2 = \frac{(1+r)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$

Demand for consumption in both periods is proportional to wage income. c_1 depends negatively on r if $\sigma > 1$, positively if $\sigma < 1$. Consumption in period 2 always depends positively on r, because the income and substitution effects work in the same direction. The easiest way to see this more formally is to rewrite the expression for c_2 as

$$c_2 = \frac{1}{(1+\rho)^{\sigma}(1+r)^{-\sigma} + (1+r)^{-1}} w$$

An increase in r reduces both terms in thee numerator and, hence, increases the fraction.

A higher ρ shifts consumption towards period 1 (increases c_1, reduces c_2).

Details on finding the solution for c_1 and c_2:

We start from (see above):

$$c_1 + \frac{1}{1+r}\left(\frac{1+r}{1+\rho}\right)^{\sigma} c_1 = w$$

Multiply with $(1+\rho)^{\sigma}$ on both sides:

$$c_1(1+\rho)^{\sigma} + c_1(1+r)^{\sigma-1} = w(1+\rho)^{\sigma}$$

Solve!

$$c_1 = \frac{(1+\rho)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$

From the first order condition:

$$c_2 = c_1\left(\frac{1+r}{1+\rho}\right)^{\sigma} = \frac{(1+\rho)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$

or,

$$c_2 = \frac{(1+r)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$