- Fully funded system: government saves the pension-tax revenue, so no effect on aggregate savings (increased government savings h matches reduction in private savings)

$$S_t = b + h + k$$

  - Note: no need for such pension system unless some people are irrational (rational households can save on their own)

7 Decisions under uncertainty

Purpose of lecture:

1. study decisions under uncertainty
2. Understand the permanent income hypothesis

7.1 Expected utility

- Define a *state of the world* as one particular realization of uncertainty (e.g., rain or sun tomorrow)
- Make two key assumptions:
  1. Suppose households have concave preferences over consumption, $u(c)$, which are separable between different states of the world. That is, utility in one state of the world does not depend on events that did not happen.
  2. Assume that households fully understand the probabilities of the risk they face
- Then households make decisions under uncertinaty as if they maximize *expected utility*:

$$\max E\{u(c)\}$$

- Example: Compare two lotteries A and B. Lottery A gives c_A with probability p_A and zero otherwise. Lottery B gives c_B with probability p_B and zero otherwise. What lottery will be preferred? Houshold will choose A if

$$p_A u(c_A) + (1 - p_A) u(0) > p_B u(c_B) + (1 - p_B) u(0)$$

- Jensen's inequality imply that housholds prefer certainty whenever u is strictly concave (FIGURE):

$$u(E\{c\}) > E\{u(c)\}$$

- In most cases such behavior makes sense (e.g., purchase of fire insurance)

- Deviations from expected utility: Preferences for gambling and Allais paradox. These deviations imply one of two things: (1) housholds often misunderstands or overweights probabilities close to zero or one, or (2) preferences do not satisfy expected utility theory, for example because they have *prospect theory* preferences)

7.2 A two-period consumption-savings problem

- Consider the following problem:

$$
\begin{aligned}
& \max E_1 \left\{ u\left(c_1\right) + \beta u\left(c_2\right) \right\} \\
& \text{subject to} \\
c_1 &= 1 - a_2 \\
c_2 &= \tilde{w}_2 + (1+r)\, a_2,
\end{aligned}
$$

where income in period 2, $\tilde{w}_2$, is uncertain:

$$
\tilde{w}_2 = \begin{cases} 1+\varepsilon & \text{with prob. } p \\ 1-\varepsilon & \text{with prob. } 1-p \end{cases}
$$

- Substitute the budget constraints into the utility function and reqrite the problem:

$$
\begin{aligned}
& \max_{a_2} E_1 \left\{ u\left(1-a_2\right) + \beta u\left(\tilde{w}_2 + (1+r)\, a_2\right) \right\} \\
= & \max_{a_2} \sum_{i=1}^{2} p_i \left\{ u\left(1-a_2\right) + \beta u\left(\tilde{w}_2 + (1+r)\, a_2\right) \right\} \\
= & \max_{a_2} \left\{ u\left(1-a_2\right) + p\beta u\left(1+\varepsilon + (1+r)\, a_2\right) \right. \\
& \left. + (1-p)\, \beta u\left(1-\varepsilon+(1+r)\, a_2\right) \right\}
\end{aligned}
$$

Differentiate w.r.t. a_2:

$$
\begin{aligned}
0 &= -u'\left(1-a_2\right) + \beta(1+r) \cdot \\
& \quad \left[pu'\left(1+\varepsilon+(1+r)\, a_2\right) + (1-p)\, u'\left(1-\varepsilon+(1+r)\, a_2\right) \right] \\
& \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
u'\left(1-a_2\right) &= \beta(1+r) \cdot \left[pu'\left(1+\varepsilon+(1+r)\, a_2\right) + (1-p)\, u'\left(1-\varepsilon+(1+r)\, a_2\right) \right] \\
&= E_1 \left\{ (1+r) \cdot \beta u'\left(\tilde{w}_2 + (1+r)\, a_2\right) \right\} \\
& \Rightarrow \\
u'\left(c_1\right) &= E_1 \left\{ (1+r) \cdot \beta u'\left(c_2\right) \right\} \qquad (13)
\end{aligned}
$$

Comments:

- This is the Euler equation under uncertainty.

– Note that solving this problem did require *rational expectations* (but not necessarily *perfect foresight*). Rational expectations requires that the household knows the future probabilities, not the realized outcomes.

- Dynamics: Suppose there are more periods, so the problem is

$$\begin{aligned} & \max E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \\ c_t + a_{t+1} &= \tilde{w}_t + (1+r)\, a_t \text{ for all } t \geq 0 \end{aligned}$$

Write this problem in value function form:

$$\begin{aligned} V(a_t, w_t) &= \max \{u(c_t) + E_t V(a_{t+1}, w_{t+1})\} \\ & \text{subject to} \\ c_t + a_{t+1} &= w_t + (1+r)\, a_t, \end{aligned}$$

where the value function V is the expected discounted utility, and a and w are the state variables. Given the function V, consmption can be expressed as a function of the same state variables: $C(a, w)$. This is the *policy rule* for consumption. It can be shown that a condition for optimality is the Euler equation, $u'(c_t) = E_t \{(1+r) \cdot \beta u'(c_{t+1})\}$, so the policy rule must satisfy the following Euler equation for any combination of the state variables (a, w):

$$u'(C(a_t, w_t)) = E_t \{(1+r) \cdot \beta u'(C(a_{t+1}, w_{t+1}))\}.$$

- *Detour on asset pricing*: note that the Euler equation determines the price of a (riskless) bond q by rewriting equation (13) as follows:

$$q \equiv \frac{1}{1+r} = E_1 \left\{ \beta \frac{u'(c_2)}{u'(c_1)} \right\}$$

To get this, note that c_1 is known at time $t = 1$, so it is allowed to divide inside the expectation operator. This equation can also be used to find the period-1 price p_1 of any asset (whose price and dividends are stochastic and given by $\tilde{p}_2$ and $\tilde{d}_2$ next period):

$$\begin{aligned} p_1 \cdot u'(c_1) &= E_1 \left\{ \beta u'(c_2) \cdot \left(\tilde{p}_2 + \tilde{d}_2 \right) \right\} \\ &\Rightarrow \\ p_1 &= E_1 \left\{ \underbrace{\beta \frac{u'(c_2)}{u'(c_1)}}_{\text{stochastic discount factor}} \cdot \left(\tilde{p}_2 + \tilde{d}_2 \right) \right\} \end{aligned}$$

... this equation is what the field of *asset pricing* in finance is all about.

7.3 Permanent income hypothesis: a special case

- We will now consider a special case of decisions under uncertainty, namely when preferences are linear-quadratic:

$$u(c) = c - \frac{a}{2}c^2$$

Note that in this case, marginal utility is given by

$$u'(c) = 1 - a \cdot c$$

- Linear-quadratic preferences imply the following Euler equation:

$$\begin{aligned} 1 - ac_t &= E_t \{(1+r) \cdot \beta (1 - ac_{t+1})\} && (14) \\ &= (1+r)\beta - (1+r)\beta a E_t \{c_{t+1}\} \\ &\Rightarrow \\ c_t &= \frac{1-(1+r)\beta}{a} + (1+r)\beta \cdot E_t \{c_{t+1}\} && (15) \end{aligned}$$

- Suppose, for simplicity, that r is such that $(1+r)\beta = 1$, i.e., there is no intertemporal motive to save. In this case the Euler equation (15) becomes

$$c_t = E_t \{c_{t+1}\},$$

so the household wants to hold *expected* consumption constant.

- Recall that the budget constraint has to hold with equality. Since Ponzi schemes are ruled out ($1 + r = 1/\beta > 1$), the discounted value of consumptoin must equal discounted income:

$$(1+r) a_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}$$

Take the expected value on both sides to obtain that consumption must equal expected income:

$$\begin{aligned} (1+r) a_0 + E_0 \left\{ \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} \right\} &= E_0 \left\{ \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \right\} \\ &= \sum_{t=0}^{\infty} \frac{E_0 c_t}{(1+r)^t} \end{aligned}$$

The *law of iterated expectations* says that

$$E_0 \{x_t\} = E_0 \{E_1 \{x_t\}\},$$

which implies that

$$E_0 c_t = E_0 \{E_1 c_t\} = ... = E_0 \{E_1 \{E_2 ... \{E_{t-1} c_t\}\}\} = c_0.$$

The Euler equation can then be rewritten as

$$\begin{aligned}(1+r)\,a_0 + E_0\left\{\sum_{t=0}^{\infty}\frac{w_t}{(1+r)^t}\right\} &= \sum_{t=0}^{\infty}\frac{E_0 c_t}{(1+r)^t} \\ &= c_0\left(\sum_{t=0}^{\infty}\frac{1}{(1+r)^t}\right) = \frac{c_0}{r}\end{aligned}$$

which implies that consumption is given by

$$c_0 = r\cdot\left[(1+r)\,a_0 + E_0\left\{\sum_{t=0}^{\infty}\frac{w_t}{(1+r)^t}\right\}\right],$$

Conclusion: optimal consuption is to consume a constant share r of expected lifetime wealth W. This is the *permanent income hypothesis:* $c = rW$

- By the definition of expectations, we can always write

  $$c_t = E_{t-1}\{c_t\} + \varepsilon_t,$$

  where ε_t is a stochastic variable with

  $$E_{t-1}\{\varepsilon_t\} = 0$$

  Since $c_t = E_{t-1}\{c_t\}$, we can then write

  $$c_t = c_{t-1} + \varepsilon_t,$$

  which is the *random-walk hypothesis* of Hall (1978).

- What is ε_t? Simplify by setting $r = 0$ and finite horizon, so that

  $$c_0 = \frac{1}{T-1}\cdot\left[a_0 + E_0\left\{\sum_{t=0}^{T} w_t\right\}\right]$$

  Compute c_1

  $$\begin{aligned}c_1 &= \frac{1}{T-1}\cdot\left[a_1 + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\ &= \frac{1}{T-1}\cdot\left[-c_0 + a_0 + w_0 + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\ &= \frac{1}{T-1}\cdot\left[-c_0 + a_0 + w_0 + E_0\left\{\sum_{t=1}^{T} w_t\right\} - E_0\left\{\sum_{t=1}^{T} w_t\right\} + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\ &= \frac{1}{T-1}\cdot\left[-c_0 + Tc_0 - E_0\left\{\sum_{t=1}^{T} w_t\right\} + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\ &= c_0 + \frac{1}{T-1}\cdot\left[E_1\left\{\sum_{t=1}^{T} w_t\right\} - E_0\left\{\sum_{t=1}^{T} w_t\right\}\right] \\ &\equiv c_0 + \varepsilon_1\end{aligned}$$

Here, ε_1 is the innovation in permanent income. Note:

– A permanent increase in period $t = 1$ which was unexpected in period $t = 0$: $w_t = w_0 + \Delta$ (and no further change in w_t) would give a one-for-one increase in consumption:

$$\begin{aligned} c_1 &= c_0 + \frac{1}{T-1} \cdot \left[E_1 \left\{ \sum_{t=1}^{T} (w_0 + \Delta) \right\} - E_0 \left\{ \sum_{t=1}^{T} w_0 \right\} \right] \\ &= c_0 + \frac{1}{T-1} \cdot \sum_{t=1}^{T} \Delta = c_0 + \frac{T}{T-1} \cdot \Delta \end{aligned}$$

– A transitory (one-period) increase in period $t = 1$ which was unexpected in period $t = 0$: $w_1 = w_0 + \Delta$ (and $w_t = w_0$ thereafter) would give only a small increase in consumption:

$$\begin{aligned} c_1 &= c_0 + \frac{1}{T-1} \cdot \left[E_1 \left\{ w_0 + \Delta + \sum_{t=2}^{T} w_0 \right\} - E_0 \left\{ \sum_{t=1}^{T} w_0 \right\} \right] \\ &= c_0 + \frac{1}{T-1} \Delta \end{aligned}$$

- Note: linear-quadratic preferences exhibit certainty equivalence, in the sense that rise does not matter for savings.
- Empirical tests of the random-walk hypothesis using houshold-level data:
  - Souleles (1999), Johnsen et al. (2003), and others show that houshold-level consumption responds to (small) predictable income changes (for example tax rebates that are announced long time in advance). For example, the Bush tax rebates had big effects on consumption. This is inconsistent with PIH.
  - Using household-level data, Paxson (1993) and Hsieh (2003) show that for large predicted income changes (+/- 10%), there is no response in consumption (which is consistent with PIH).
  - How can this be true? Potential resolution: many housholds may be locked into a mortgage and may be liquidity constrained in terms of temporary expenditurs. It may be costly to change the mortgage, so one would pay such a cost (and potentially get less constrained) only if the future income change is sufficiently large. Or perhaps not all housholds pay attention.

8 Consumption-based asset pricing

Purpose of lecture:

1. Explore the asset-pricing implications of the neoclassical model
2. Understand the pricing of insurance and aggregate risk
3. Understand the quantitative limitations of the model

8.1 The fundamental asset pricing equation

- Consider and economy where people live for two periods ...

$$\begin{aligned} & \max_{\{c_t, c_{t+1}, a_{t+1}\}} \{u(c_t) + \beta E_t u(c_{t+1})\} \\ & \text{subject to} \\ y_t &= c_t + p_t a_{t+1} \\ c_{t+1} &= y_{t+1} + (p_{t+1} + d_{t+1}) a_{t+1}, \end{aligned}$$

  where y_t is income in period t, p_t is the (ex-dividend) price of the asset in period t, and d_t is the dividend from the asset in period t.

- Substitute the two constraints into the utility function:

$$\max_{a_{t+1}} \{u(y_t - p_t a_{t+1}) + \beta E_t u(y_{t+1} + (p_{t+1} + d_{t+1}) a_{t+1})\}$$

  and differentiate w.r.t. how much of the asset to purchase, a_{t+1}:

$$\begin{aligned} 0 &= -p_t \cdot u'(y_t - p_t a_{t+1}) + \beta E_t \{u'(y_{t+1} + (p_{t+1} + d_{t+1}) a_{t+1}) \cdot (p_{t+1} + d_{t+1})\} \\ & \Rightarrow \\ p_t \cdot u'(c_t) &= \beta E_t \{u'(c_{t+1}) \cdot (p_{t+1} + d_{t+1})\}. \end{aligned}$$

- Interpretation: the left-hand side is the marginal cost of purchasing one additional unit of the asset, in utility terms (so price * marginal utility), while the right-hand side is the expected marginal gain in utility terms (i.e., next-period marginal utility time price+dividend).

- Rewrite to get the fundamental asset-pricing equation:

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot (p_{t+1} + d_{t+1}) \right\},$$

  where the term $\frac{\beta u'(c_{t+1})}{u'(c_t)}$ is the (stochastic) discount factor (or the "pricing kernel").

- **Key insight #1**: Suppose the household is not borrowing constrained and suppose the household has the opportunity to purchase an asset. Then this equation – and the consumption process for this household – can be used to price that asset. This applies to *any* asset and to the consumption procvess for *any* household.

- What goes wrong if the household is borrowing constrained? The problem is that if the houshold is constrained in period t. Consider, for example, a case when the household would like to borrow so as to increase current consumption, but is not allowed to do so. Then marginal utility in period t is very high and the Euler equation (for a riskless bond that pays one unit of consumption for sure next period) becomes an inequality:

$$u'(c_t) > \frac{1}{q_t}\beta E_t\{u'(c_{t+1})\}$$

  - Intuition: the household feels that the bond is very expensive (i.e., that the interest rate is very low), so it would like to sell bonds (i.e., borrow from the bank), but the bank does not allow the houshold to do so. Clearly, the consumption stream of this household cannot be used to price the asset.

8.2 Which assets are expensive?

- Define the return of an asset i as

$$1 + r^i_{t+1} = \frac{p^i_{t+1} + d^i_{t+1}}{p^i_t},$$

so the asset-pricing equation can be rewritten as

$$1 = E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot \left(1 + r^i_{t+1}\right)\right\}$$

- Compare the price of a risky asset with that of a safe asset (i.e., a bond with a safe return $\bar{r}_{t+1}$):

$$\begin{aligned}
1 &= E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot \left(1 + r^i_{t+1}\right)\right\} = E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\} + E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot r^i_{t+1}\right\} \\
1 &= E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot (1 + \bar{r}_{t+1})\right\} = E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\} \cdot (1 + \bar{r}_{t+1}),
\end{aligned}$$

where the last equation follows from the fact that $\bar{r}_{t+1}$ is riskfree. Combine these equations to obtain:

$$\begin{aligned}
E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot r^i_{t+1}\right\} &= E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\} \cdot \bar{r}_{t+1} \\
&\Rightarrow \\
E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)} \cdot \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\} &= 0, \qquad (16)
\end{aligned}$$

where the term $\left(r_{t+1}^{i}-\bar{r}_{t+1}\right)$ is the (stochastic) *excess return* on the risky asset.

- Recall that the formula for covariance between two stochastic variables X and Y is

$$\begin{aligned} cov\left(X,Y\right) &= E\left\{X\cdot Y\right\}-E\left(X\right)\cdot E\left(Y\right) \\ &\Rightarrow \\ E\left\{X\cdot Y\right\} &= E\left(X\right)\cdot E\left(Y\right)+cov\left(X,Y\right) \end{aligned}$$

- Multiply equation (16) by $u'\left(c_t\right)/\beta$ on both sides, and use the covariance formula:

$$\begin{aligned} 0 &= E_t\left\{u'\left(c_{t+1}\right)\cdot\left(r_{t+1}^{i}-\bar{r}_{t+1}\right)\right\} \\ &= cov\left\{u'\left(c_{t+1}\right),\left(r_{t+1}^{i}-\bar{r}_{t+1}\right)\right\}+E_t\left\{u'\left(c_{t+1}\right)\right\}\cdot E_t\left(r_{t+1}^{i}-\bar{r}_{t+1}\right) \\ &= cov\left\{u'\left(c_{t+1}\right),\left(r_{t+1}^{i}-\bar{r}_{t+1}\right)\right\}+E_t\left\{u'\left(c_{t+1}\right)\right\}\cdot\left(E_t\left(r_{t+1}^{i}\right)-\bar{r}_{t+1}\right) \end{aligned} \tag{17}$$

The term $E_t\left(r_{t+1}^{i}\right)-\bar{r}_{t+1}$ is the "risk premium", i.e., the expected excess return, or the expected return on asset i relative to the return on the riskless bond.

- Make two key assumptions:

  1. Suppose the economy nis a representative-agent economy, so that each household's consumption is a constant share of the aggregate consumption ($c_t = C_t$)
  2. Suppose (for simplicity) that the utility function is quadratic, i.e., that

$$\begin{aligned} u\left(c\right) &= c-a\frac{c^2}{2} \\ &\Rightarrow \\ u'\left(c\right) &= 1-ac \end{aligned}$$

  Insert this expression for u' into equation (17):

$$\begin{aligned} 0 &= cov\left\{\left(1-aC_{t+1}\right),\left(r_{t+1}^{i}-\bar{r}_{t+1}\right)\right\}+E_t u'\left(C_{t+1}\right)\cdot\left(E_t\left(r_{t+1}^{i}\right)-\bar{r}_{t+1}\right) \\ &= -a\cdot cov\left\{C_{t+1},\left(r_{t+1}^{i}-\bar{r}_{t+1}\right)\right\}+E_t u'\left(C_{t+1}\right)\cdot\left(E_t\left(r_{t+1}^{i}\right)-\bar{r}_{t+1}\right) \end{aligned} \tag{18}$$

- Consider three cases:

  1. The market portfolio: Suppose asset M is the market portfolio (i.e., the whole stock market). Note that both the stock market and aggregate consumption increase when times are good, so the return on stocks is highly correlated with consumption:

$$cov\left\{C_{t+1},\left(r_{t+1}^{M}-\bar{r}_{t+1}\right)\right\}>0.$$

Equation (18) then implies that

$$E_t u'(C_{t+1}) \cdot \left(E_t\left(r^i_{t+1}\right) - \bar{r}_{t+1}\right) > 0,$$

so the return on the market portfolio must be higher than the safe return (since $u'(c) > 0$):

$$E_t\left(r^i_{t+1}\right) > \bar{r}_{t+1}$$

Key message: the *risk premium* on the market portolio is positive because the aggregate stock market is correlated with aggregate consumption.

2. An asset with non-systematic risk: Suppose the asset i is risky, but the return is completely uncorrelated with aggregate consumption:

$$cov\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\} = 0.$$

Equation (18) then implies that

$$E_t u'(C_{t+1}) \cdot \left(E_t\left(r^i_{t+1}\right) - \bar{r}_{t+1}\right) = 0,$$

which again implies that this asset has the same expected return as the safe bond, even if it is risky:

$$E_t\left(r^i_{t+1}\right) = \bar{r}_{t+1},$$

Key message: there is zero premium for holding idiosyncratic risk (i.e., risk which is uncorrelated with aggregate consumption).

3. Insurance: Suppose the asset i is risky, but the return is negatively correlated with aggregate consumption:

$$cov\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\} < 0,$$

so the return is high precicely when consumption is low. This is an example of an asset which serves as *insurance*. Equation (18) then implies that

$$\begin{aligned} E_t u'(C_{t+1}) \cdot \left(E_t\left(r^i_{t+1}\right) - \bar{r}_{t+1}\right) &< 0 \\ &\Rightarrow \\ E_t\left(r^i_{t+1}\right) &< \bar{r}_{t+1}, \end{aligned}$$

so this asset has a *lower* return than the safe bond (i.e., a negative risk premium). Key message: households are willing to *pay* a premium in order to get insurance.

8.3 Consumption-based CAPM

- Rewrite equation (18) as follows:

$$E_t\left(r^i_{t+1}\right) - \bar{r}_{t+1} = \frac{a \cdot cov\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\}}{E_t\left[u'\left(C_{t+1}\right)\right]},$$

  so the expected premium return on the market portfolio is

$$E_t\left(r^M_{t+1}\right) - \bar{r}_{t+1} = \frac{a \cdot cov\left\{C_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\}}{E_t\left[u'\left(C_{t+1}\right)\right]}.$$

- Rewrite the expected return on asset i as

$$\begin{aligned}
E_t\left(r^i_{t+1}\right) &= \frac{cov\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\}}{cov\left\{C_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\}} \frac{a \cdot cov\left\{C_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\}}{E_t\left[u'\left(C_{t+1}\right)\right]} + \bar{r}_{t+1} \\
&= \frac{cov\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\}}{cov\left\{C_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\}} \left[E_t\left(r^M_{t+1}\right) - \bar{r}_{t+1}\right] + \bar{r}_{t+1} \\
&= \frac{corr\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\} std\left(C_{t+1}\right) std\left(r^i_{t+1}\right)}{corr\left\{C_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\} std\left(C_{t+1}\right) std\left(r^M_{t+1}\right)} \left[E_t\left(r^M_{t+1}\right) - \bar{r}_{t+1}\right] + \bar{r}_{t+1} \\
&= \frac{std\left(r^i_{t+1}\right)}{std\left(r^M_{t+1}\right)} \frac{corr\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\}}{corr\left\{C_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\}} \left[E_t\left(r^M_{t+1}\right) - \bar{r}_{t+1}\right] + \bar{r}_{t+1} \\
&\equiv std\left(r^i_{t+1}\right) \cdot BETA_i \cdot \left[E_t\left(r^M_{t+1}\right) - \bar{r}_{t+1}\right] + \bar{r}_{t+1},
\end{aligned}$$

  where the "consumption $BETA_i$" is defined as

$$BETA_i \equiv \frac{1}{std\left(r^M_{t+1}\right)} \frac{corr\left\{C_{t+1}, \left(r^i_{t+1} - \bar{r}_{t+1}\right)\right\}}{corr\left\{C_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\}}$$

  Interpretation: the term $BETA_i$ for an asset i reveals the risk premium of additional risk of this asset.

8.4 Equity premium puzzle

- Go back to equation (16),

$$E_t\left\{m_{t+1} \cdot \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right\} = 0,$$

  where the stochastic discount factor m_{t+1} is

$$m_{t+1} = \frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}.$$

  Using the formula for the covariance, rewrite this as

$$\begin{aligned}
0 &= E_t\left\{m_{t+1}\right\} \cdot E_t\left\{r^M_{t+1} - \bar{r}_{t+1}\right\} + cov\left(m_{t+1}, \left(r^M_{t+1} - \bar{r}_{t+1}\right)\right) \\
&= E_t\left\{m_{t+1}\right\} \cdot E_t\left\{r^M_{t+1} - \bar{r}_{t+1}\right\} + corr\left(m_{t+1}, r^M_{t+1}\right) \cdot std\left(m_{t+1}\right) \cdot std\left(r^M_{t+1}\right) \\
&\Rightarrow \\
std\left(m_{t+1}\right) &= -E_t\left\{m_{t+1}\right\} \cdot \frac{E_t\left\{r^M_{t+1} - \bar{r}_{t+1}\right\}}{std\left(r^M_{t+1}\right)} \frac{1}{corr\left(m_{t+1}, r^M_{t+1}\right)}
\end{aligned}$$

Recall that for the safe asset we have

$$\frac{1}{1+\bar{r}_{t+1}} = E_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\} = E_t\{m_{t+1}\},$$

so we can rewrite the equation above as

$$std(m_{t+1}) = -\frac{1}{corr\left(m_{t+1}, r_{t+1}^M\right)}\frac{1}{1+\bar{r}_{t+1}}\frac{E_t\left\{r_{t+1}^M - \bar{r}_{t+1}\right\}}{std\left(r_{t+1}^M\right)}$$

- Note that $corr\left(m_{t+1}, r_{t+1}^M\right) < 0$ (since aggregate consumption and r_{t+1}^M are positively correlated) and that by definition,

$$-corr\left(m_{t+1}, r_{t+1}^M\right) \leq 1.$$

Therefore, the smallest possible value of the term $-\frac{1}{corr\left(m_{t+1}, r_{t+1}^M\right)}$ is one:

$$-\frac{1}{corr\left(m_{t+1}, r_{t+1}^M\right)} \geq 1.$$

This implies a (Hansen-Jaganathan) bound on the variability of m_{t+1}:

$$std(m_{t+1}) \geq \frac{1}{1+\bar{r}_{t+1}}\frac{E_t\left\{r_{t+1}^M - \bar{r}_{t+1}\right\}}{std\left(r_{t+1}^M\right)} \tag{19}$$

- The term $\frac{E_t\left\{r_{t+1}^M - \bar{r}_{t+1}\right\}}{std\left(r_{t+1}^M\right)}$ is the Sharpe ratio (after the Nobel Laureate William F. Sharpe), i.e., the "return per unit of risk".

- Data: The Sharpe ratio is typically around 40% on an annual basis in developed countries, while the annual safe interest rate has been close to zero on average.

  - Thus, the standard deviation of the stochastic discount factor should be around 40%

- Suppose the utility function exhibits constant relative risk aversion, i.e.,

$$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma},$$

so the discount factor becomes

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{\gamma},$$

which is close to 1 on average (at least when the time period is short). The standard deviation of m is therefore approximately

$$\begin{aligned} std\left\{\log\left(\beta\left(\frac{C_{t+1}}{C_t}\right)^{\gamma}\right)\right\} &= std\left\{\log(\beta) + \gamma\log\left(\frac{C_{t+1}}{C_t}\right)\right\} \\ &= \gamma \cdot std\left\{\log\left(\frac{C_{t+1}}{C_t}\right)\right\} \end{aligned}$$

Using U.S. data, the volatility of consumption growth is about

$$std\left\{\log\left(\frac{C_{t+1}}{C_t}\right)\right\} \approx 3\%.$$

Equation (19) then implies that γ must be at least $\gamma = 40/3 \approx 13$, which is very large.

- Using other approximations and bounds, it is straightforward to show that in this model, the risk aversion must be at least 50 in order to account for a risk premium of 6%, when the standard deviation of consumption growth is just 3% and the variability of the stock market is $std\left(r^M\right) = 16\%$. This is the *equity premium puzzle*.

- Note that estimates of the risk aversion (using micro data) implies a risk aversion somewhere in the range of $\gamma \in [1, 5]$., which is MUCH lower than 50.

- For example, with a risk aversion of 25, a household who is offered a 50/50 change of a gain or loss of 20% of lifetime consumption, would prefer to rather take a 17% loss for sure.