Dynamic analysis: linearization and impulse responses

Lecture 13, ECON 4310

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September 30, 2013
Very very early

So far..

Some of the (many) things you have learned so far in this course are:

1. The set-up of the basic neoclassical model
2. An equivalent formulation using the social planner's problem
3. Analysis of steady state

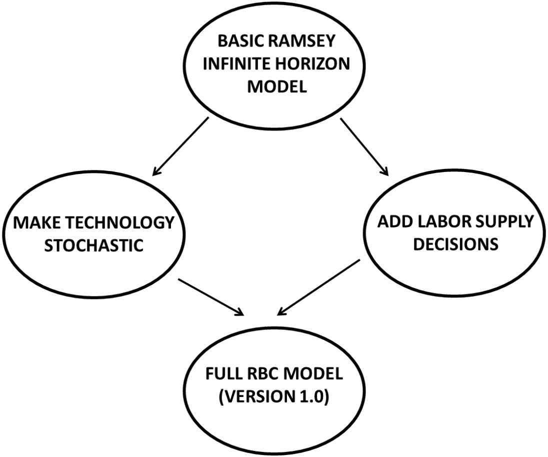
These three points are covered by chapters 3-5 in Krueger's notes.

Today's lecture

Today we move on to see how the models can be solved (chapter 6 of Krueger) as well as using our first *stochastic* model (chapter 10).

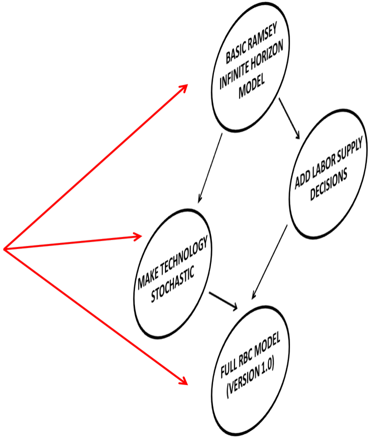
1. Quick review of basic model
2. Example of an analytical solution
3. Concept of linearization
4. Linearizing the basic model
5. Lineraizing the stochastic version of the basic model
6. Impulse-response functions

Roadmap


BASIC RAMSEY INFINITE HORIZON MODEL
MAKE TECHNOLOGY STOCHASTIC
ADD LABOR SUPPLY DECISIONS
FULL RBC MODEL (VERSION 1.0)

Behind the roadmap

CHALLENGE IN ALL THE MODELS: HOW TO GO FROM FIRST-ORDER CONDITIONS TO AN ACTUAL SOLUTION?



Basic model

We are considering the social planner's problem:

$$\max_{\{c_s, k_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^s u(c_s)$$

$$s.t.$$

$$c_t + k_{t+1} = A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t$$

$$c_t \geq 0$$

$$k_{t+1} \geq 0$$

$$0 \leq n_t \leq 1$$

with $k_t > 0$ given. Can simplify by setting $n_t = 1$ and ignore $c_t \geq 0$ and $k_{t+1} \geq 0$ under 'normal' assumptions. Note that the full RBC model will have

- Productivity (A) being stochastic
- Labor supply entering the utility function

Basic model II

Optimum is characterized by the Euler equation:

$$u'(c_t) = \beta[1 - \delta + \alpha A k_{t+1}^{\alpha-1}]u'(c_{t+1})$$

the resource constriant:

$$c_t + k_{t+1} = A k_t^{\alpha} + (1 - \delta)k_t$$

as well as a transversality condition:

$$\lim_{T\to\infty} \beta^T u'(c_T) = 0$$

Basic model III

Intuition behind the transversality condition? **Simple intuition:** It replaces a $k_{T+1} = 0$ restriction for the finite-period case. **More complicated answer:** Start out with the Euler equation:

$$\begin{aligned} u'(c_t) &= \beta[1-\delta+r_{t+1}]u'(c_{t+1}) \\ &= \beta(r_{t+1}-\delta)u'(c_{t+1}) + \beta\left\{\beta[1-\delta+r_{t+2}]u'(c_{t+2})\right\} \\ &= \ldots \\ &= \sum_{s=t+1}^{T} \beta^{s-t}(r_s-\delta)u'(c_s) + \beta^T u'(c_{T+1}) \end{aligned}$$

We see that the transversality condition restricts the last term on the RHS to converge to zero. **Note typo in first version of slides**

Basic model IV

Hence under

$$\lim_{T\to\infty} \beta^T u'(c_T) = 0$$

we have

$$u'(c_t) = \sum_{s=t+1}^{\infty} \beta^{s-t}(r_s - \delta)u'(c_s)$$

If this condition fails to be satisfied the agent could increase lifetime utility by saving one unit extra of capital and re-investing it in every period forever (consuming the dividends).

Basic model V

Steady state? With no growth in technology or population, the steady state requires $k_t = k_{t+1}$ and $c_t = c_{t+1}$. In the Euler equation that gives:

$$u'(c^*) = \beta[1 - \delta + \alpha A {k^*}^{\alpha - 1}]u'(c^*)$$

or

$$k^* = \left(\frac{\rho + \delta}{\alpha A}\right)^{\frac{1}{\alpha - 1}}$$

where $\rho = \frac{1}{\beta} - 1$ is the discount *rate*. Steady state consumption follows from the resource constraint

$$c^* = A{k^*}^{\alpha} - \delta k^*$$

Basic model VI

With this we have

- Conditions that must be satisfied in optimum
- A description of the steady state

What remains is to answer: How do we find the set $\{c_s^*, k_{s+1}^*\}_{s=t}^{\infty}$ that actually solves the problem?

Solving the model: Analytical solution

In general: Not easy to find an analytical solution for the problem. But we can simplify the model to a case where we can find one.

Solving the model: Analytical solution II

Assume that

$$u(C) = \log C$$

and also simplify by setting $\delta = 1$. The transversality condition is satisfied in steady state (c_t constant) as long as $\beta < 1$, so let us ignore that condition. Combining the Euler equation with the resource constraint, we have:

$$\frac{1}{Ak_t^\alpha - k_{t+1}} = \frac{\beta \alpha A k_{t+1}^{\alpha-1}}{Ak_{t+1}^\alpha - k_{t+2}}$$

The sequence of capital stocks, $\{k_{s+1}\}_{s=t}^{\infty}$, that satisfies this condition for all t is the solution to our problem. How to find it?

Solving the model: Analytical solution III

We use guess and verify. Let us guess that the solution is a constant savings rate, namely

$$k_{t+1} = sAk_t^{\alpha}$$

for some value of s. Insert that into our condition to find

$$\frac{1}{(1-s)Ak_t^{\alpha}} = \frac{\beta\alpha Ak_{t+1}^{\alpha-1}}{(1-s)Ak_{t+1}^{\alpha}}$$

which can be simplified to give

$$k_{t+1} = \beta\alpha Ak_t^{\alpha}$$

Hence if $s = \beta\alpha$, we see that this is indeed the solution. For any initial capital stock k_0, setting $k_{t+1} = \beta\alpha Ak_t^{\alpha}$ will ensure that the Euler condition, the resource constraint, and the transversality condition hold. Hence it is the solution.

Linearization

Fine. The solution to our problem under log utility and full depreciation turned out to be a constant savings rate. **But analytical solutions like that are usually much more difficult to find.** We will instead most of the time apply linearization techniques.

Linearization II

[Remember that dynamic programming is another alternative, but we put most emphasis on linearization in this class]

Linearization III

Main point with linearization: To solve the model as an approximation around its steady state. Hence we assume that the steady state is a relevant concept, and look at the dynamics of the model when the economy faces small departures from steady state.

Linearization IV

What does linearization entail? We will do first-order Taylor approximations. This means utilizing the Taylor approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

for a point a. For $x = a$, this holds trivially. It is a good approximation for values of x "close to" a. [But there is hardly any discussion of what is close enough]. For a multivariate function $f(x, y)$ we can approximate it around the point (a, b) using

$$f(x, y) \approx f(a, b) + \frac{\partial f(a, b)}{\partial x}(x - a) + \frac{\partial f(a, b)}{\partial y}(y - b)$$

Linearization V

Another trick we will use is to define $\hat{x}_t = \log x_t - \log x_{ss}$, i.e. the log deviation of x_t from steady state. This often comes in handy because log differences are approximately the percentage difference:

$$\log x_t - \log x^* = \log\left(\frac{x_t}{x^*}\right) = \log\left(\frac{x_t - x^*}{x^*} + 1\right) \approx \frac{x_t - x^*}{x^*}$$

One way to use that approximation, is when we want to rewrite a levels variable in its percentage deviation. First do the following:

$$x_t = e^{\log x_t} = x^* e^{\log x_t - \log x^*} = x^* e^{\hat{x}_t}$$

Then Taylor approximate the last expression around $\hat{x}_t = 0$ (i.e. the steady state):

$$e^{\hat{x}_t} \approx e^0 + e^0(\hat{x}_t - 0) = 1 + \hat{x}_t$$

Hence:

$$x_t \approx x^*(1 + \hat{x}_t)$$

Linearizing the model

We are ready to linearize the model. We know that the solution is characterized by the Euler equation:

$$u'(c_t) = \beta[1 - \delta + \alpha A k_{t+1}^{\alpha-1}]u'(c_{t+1})$$

and the resource constraint:

$$c_t + k_{t+1} = A k_t^{\alpha} + (1 - \delta) k_t$$

(we ignore the transversality since we are approximating the solution close to steady state)

Linearizing the model II

Start with the resource constraint. One element at the time:

$$
\begin{aligned}
c_t &\approx c^*(1+\hat{c}_t) \\
k_{t+1} &\approx k^*(1+\hat{k}_{t+1}) \\
Ak_t^\alpha &\approx Ak^{*\alpha} + \alpha Ak^{*\alpha-1}(k_t - k^*) \\
&\approx Ak^{*\alpha} + \alpha Ak^{*\alpha-1}k^*\hat{k}_t \\
&= Ak^{*\alpha}(1+\alpha\hat{k}_t) \\
(1-\delta)k_t &\approx (1-\delta)k^*(1+\hat{k}_t)
\end{aligned}
$$

Plug this into the resource constraint:

$$
c^*(1+\hat{c}_t) + k^*(1+\hat{k}_{t+1}) \approx Ak^{*\alpha}(1+\alpha\hat{k}_t) + (1-\delta)k^*(1+\hat{k}_t)
$$

Linearizing the model III

We simplify futher by using

$$c^* = Ak^{*\alpha} - \delta k^*$$

since then

$$c^*(1 + \hat{c}_t) + k^*(1 + \hat{k}_{t+1}) \approx Ak^{*\alpha}(1 + \alpha \hat{k}_t) + (1 - \delta)k^*(1 + \hat{k}_t)$$

simplifies to:

$$c^* \hat{c}_t + k^* \hat{k}_{t+1} = \left[1 + \alpha A k^{*\alpha - 1} - \delta\right] k^* \hat{k}_t$$

Finally, since we know that in steady state:

$$1 = \beta(1 - \delta + \alpha A k^{*\alpha - 1})$$

we get:

$$c^* \hat{c}_t + k^* \hat{k}_{t+1} = \frac{1}{\beta} k^* \hat{k}_t \qquad (1)$$

Note the slight change from first version of slides

Linearizing the model IV

The fun continues with the Euler equation. To linearize marginal utility is simple:

$$\begin{aligned} u'(c_t) &\approx u'(c^*) + u''(c^*)(c_t - c^*) \\ &\approx u'(c^*) + u''(c^*)c^*\hat{c}_t \\ &= u'(c^*)\left[1 + \frac{c^* u''(c^*)}{u'(c^*)}\hat{c}_t\right] \\ \beta(1-\delta)u'(c_{t+1}) &\approx \beta(1-\delta)u'(c^*)\left[1 + \frac{c^* u''(c^*)}{u'(c^*)}\hat{c}_{t+1}\right] \end{aligned}$$

Linearizing the model V

Then there is the last term:

$$
\begin{aligned}
&\alpha A k_{t+1}^{\alpha-1} \beta u'(c_{t+1}) \\
&\approx \alpha A {k^*}^{\alpha-1} \beta u'(c^*) + \alpha(\alpha-1) A {k^*}^{\alpha-2} \beta u'(c^*)(k_{t+1} - k^*) + \alpha A {k^*}^{\alpha-1} \beta u''(c^*)(c_{t+1} - c^*) \\
&\approx \alpha A {k^*}^{\alpha-1} \beta u'(c^*) + \alpha(\alpha-1) A {k^*}^{\alpha-2} \beta u'(c^*) k^* \hat{k}_{t+1} + \alpha A {k^*}^{\alpha-1} \beta u''(c^*) c^* \hat{c}_{t+1} \\
&= \alpha A {k^*}^{\alpha-1} \beta u'(c^*) \left[1 + (\alpha - 1)\hat{k}_{t+1} + \frac{c^* u''(c^*)}{u'(c^*)} \hat{c}_{t+1} \right]
\end{aligned}
$$

Linearizing the model VI

Let us define

$$\theta = -\frac{c^* u''(c^*)}{u'(c^*)}$$

(we return to what θ is in a minute). Combine all the approximations for the Euler equation:

$$u'(c^*)\left[1-\theta\hat{c}_t\right] = \beta(1-\delta)u'(c^*)\left[1-\theta\hat{c}_{t+1}\right] + \alpha A {k^*}^{\alpha-1}\beta u'(c^*)\left[1+(\alpha-1)\hat{k}_{t+1}-\theta\hat{c}_{t+1}\right]$$

Let $u'(c^*)$ go against each other:

$$\left[1-\theta\hat{c}_t\right] = \beta(1-\delta)\left[1-\theta\hat{c}_{t+1}\right] + \alpha A {k^*}^{\alpha-1}\beta\left[1+(\alpha-1)\hat{k}_{t+1}-\theta\hat{c}_{t+1}\right]$$

Linearizing the model VII

We can actually do more. Once more we use that in steady state:

$$1 = \beta(1 - \delta + \alpha A {k^*}^{\alpha-1})$$

so in the Euler equation:

$$\begin{aligned}
[1 - \theta\hat{c}_t] &= \beta(1-\delta)\left[1 - \theta\hat{c}_{t+1}\right] + \beta\alpha A {k^*}^{\alpha-1}\left[1 + (\alpha-1)\hat{k}_{t+1} - \theta\hat{c}_{t+1}\right] \\
-\theta\hat{c}_t + [1 - \beta(1 - \delta - \alpha A {k^*}^{\alpha-1})] &= -\beta(1 - \delta + \alpha A {k^*}^{\alpha-1})\theta\hat{c}_{t+1} + \beta\alpha A {k^*}^{\alpha-1}(\alpha-1)\hat{k}_{t+1} \\
-\theta\hat{c}_t &= -\theta\hat{c}_{t+1} + \beta\alpha A {k^*}^{\alpha-1}(\alpha-1)\hat{k}_{t+1}
\end{aligned}$$

(Note how we use $\beta(1 - \delta + \alpha A {k^*}^{\alpha-1}) = 1$ two different places).

Linearizing the model VIII

Dividing by $-\theta$ we are finished (for now) with the Euler equation:

$$\hat{c}_t = \hat{c}_{t+1} - \beta \frac{1}{\theta} \alpha A {k^*}^{\alpha-1} (\alpha - 1) \hat{k}_{t+1} \tag{2}$$

What is θ?

$$\theta = -\frac{c^* u''(c^*)}{u'(c^*)}$$

θ is the *coefficient of relative risk aversion* (evaluated in the steady state). Recall seminar 1 where we showed that for a utility function

$$u(c) = \frac{c^{1-1/\sigma}}{1 - 1/\sigma}$$

the intertemporal elasticity of substitution is σ, while the CRRA is $1/\sigma$. Hence, if we impose this utility function, then $\theta = 1/\sigma$! So what determines the impact of $\hat{k}_{t+1}$ (which influences the interest rate) on consumption is the intertemporal elasticity of substitution.

Linearizing the model IX

So, the log-linearized pair of equilibrium conditions (1) and (2) read:

$$c^* \hat{c}_t + k^* \hat{k}_{t+1} = \frac{1}{\beta} k^* \hat{k}_t$$

$$\hat{c}_t = \hat{c}_{t+1} - \beta \frac{1}{\theta} \alpha A {k^*}^{\alpha-1} (\alpha - 1) \hat{k}_{t+1}$$

Combine these two conditions to get:

$$\frac{k^*}{c^*} \left(\frac{1}{\beta} \hat{k}_t - \hat{k}_{t+1} \right) =$$

$$\frac{k^*}{c^*} \left(\frac{1}{\beta} \hat{k}_{t+1} - \hat{k}_{t+2} \right) - \beta \frac{1}{\theta} \alpha A {k^*}^{\alpha-1} (\alpha - 1) \hat{k}_{t+1}$$

THIS is a second-order *linear* difference equation which is straightforward to solve to get a solution for $\hat{k}_t$. Using that, we solve for $\hat{c}_t$, and then we have the complete solution. **Note: Slides 28-31 replace slide 28 from first version of slide set**

Solving the linear model X

Let us see how to find the solution. What we use is the *method of undetermined coefficients*. Suppose we conjecture that the solution is

$$\hat{k}_{t+1} = a\hat{k}_t$$

This would imply

$$\hat{k}_{t+2} = a\hat{k}_{t+1} = a^2\hat{k}_t$$

Let us insert for that solution in our difference equation:

$$\frac{k^*}{c^*}\left(\frac{1}{\beta}\hat{k}_t - a\hat{k}_t\right) =$$
$$\frac{k^*}{c^*}\left(\frac{1}{\beta}a\hat{k}_t - a^2\hat{k}_t\right) - \beta\frac{1}{\theta}\alpha A{k^*}^{\alpha-1}(\alpha-1)a\hat{k}_t$$

which is, after dividing by $\hat{k}_t$:

$$\frac{k^*}{c^*}\left(\frac{1}{\beta} - a\right) =$$
$$\frac{k^*}{c^*}\left(\frac{1}{\beta}a - a^2\right) - \beta\frac{1}{\theta}\alpha A{k^*}^{\alpha-1}(\alpha-1)a$$

Linearizing the model XI

That equation is really just a second-order equation for a (a is the only unknown). So it is 'easy' to find a, provided that a solution exists. The equation can re-arranged as

$$a^2 - \left[1 + \frac{1}{\beta} - \frac{c^*}{k*}(1-\alpha)\frac{\beta}{\theta}\alpha A k^{*\,\alpha-1}\right] a + \frac{1}{\beta} = 0$$

which has two solutions:

$$a_i = \frac{1}{2}\left[1 + \frac{1}{\beta} - \frac{c^*}{k*}(1-\alpha)\frac{\beta}{\theta}\alpha A k^{*\,\alpha-1}\right] + \frac{1}{2}(-1)^i\sqrt{\left[1 + \frac{1}{\beta} - \frac{c^*}{k*}(1-\alpha)\frac{\beta}{\theta}\alpha A k^{*\,\alpha-1}\right]^2 - 4\frac{1}{\beta}}$$

for $i = 1, 2$.

Linearizing the model XII

Can show that (see Krueger):

- Both roots are real
- Both roots are positive
- That $a_1 > 1$ and $a_2 < 1$

Hence we will use a_2 as our solution, since $a > 1$ would imply that we diverge from steady state. This means that the solution to our model is:

- $\hat{k}_{t+1} = a_2 \hat{k}_t$
- $\hat{c}_t = \frac{k^*}{c^*} \left[\hat{k}_t / \beta - \hat{k}_{t+1} \right]$

So for an initial value for $\hat{k}_0$, we have the full solution, and can solve for the path of consumption and capital (as percentage deviations from steady state).

Linearizing the model XIII

The procedure can be summarized as follows:

1. Start out with the conditions that must hold in optimum
2. Linearize the conditions around steady state. Gives (1) and (2)
3. Solve the implied system of linear difference equations
4. Gives you a solution for $\hat{k}_{t+1}$ and $\hat{c}_t$ for all t given an initial condition

Steps 3-4 are most often done with the aid of a computer. The most important job is therefore to derive optimality conditions and linearizing them.

Making the basic model stochastic

The first part of the lecture helps us see how to solve the model, i.e. see what the actual values of consumption and capital are. But this was for a deterministic model where we just have convergence to the steady state. Once we are "there", nothing more happens. More relevant to look at *stochastic* models where unexpected shocks are continuously pushing us away from steady state.

Making the basic model stochastic II

This is the whole fundament of modern business cycle models. Most DSGE models are solved by

- Deriving first-order conditions
- Linearizing around steady state
- Calibrating the model (topic for a future lecture)
- Solving for the paths of the endogenous variables
- Evaluating the effect of various shocks to the economy

In these models, *business cycles are caused by stochastic shocks and the economy's endogenous response to these.*

Stochastic basic model

We introduce stochasticity by making technology random. The parameter A in the basic model was referred to as technology. Let us make it a stochastic variable A_t, where

$$A_t = Ae^{z_t}$$

and z_t is an autoregressive error of order 1:

$$z_t = \rho z_{t-1} + \varepsilon_t$$

where ε_t is $N(0, \sigma^2)$ (confer lecture 9 for more about AR-processes). The point with defining A as we have done here, is to get:

$$\log A_t = \log A + z_t$$

so that log of productivity is linear in the shock. In the **non-stochastic steady state** we have $z_t = 0$ and $A_t = A$. Hence:

$$\hat{A}_t \approx z_t$$

so z_t is the percentage deviation of technology from its steady state value.

Stochastic basic model II

With stochastic technology, this gives us the following social planner's problem:

$$\max_{\{c_s, k_{s+1}\}_{s=t}^{\infty}} E_t \sum_{s=t}^{\infty} \beta^s u(c_s)$$

$$s.t.$$

$$c_t + k_{t+1} = A_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t$$

$$A_t = Ae^{z_t}$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

$$c_t \geq 0$$

$$k_{t+1} \geq 0$$

$$0 \leq n_t \leq 1$$

with $k_t > 0$ given. Can as before simplify by setting $n_t = 1$ and ignore $c_t \geq 0$ and $k_{t+1} \geq 0$ under 'normal' assumptions.

Stochastic basic model III

The conditions that we will work with now are the stochastic Euler equation:

$$u'(c_t) = \beta E_t \left\{ [1 - \delta + \alpha A_{t+1} k_{t+1}^{\alpha-1}] u'(c_{t+1}) \right\}$$

the resource constraint:

$$c_t + k_{t+1} = A_t k_t^{\alpha} + (1-\delta) k_t$$

and the definitions of A_t and z_t:

$$A_t = A e^{z_t}$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Stochastic basic model IV

What now? We can do the exact same thing as for the deterministic model: Linearize around a non-stochastic steady state to find a set of linear difference equation that can be solved to obtain a solution.

Linearizing the stochastic basic model

The biggest change compared to the deterministic model is that A_t is also a variable we need to take into account. Start with the resource constraint.

$$\begin{aligned} c_t &\approx c^*(1+\hat{c}_t) \quad (Same) \\ k_{t+1} &\approx k^*(1+\hat{k}_{t+1}) \quad (Same) \\ (1-\delta)k_t &= (1-\delta)k^*(1+\hat{k}_t) \quad (Same) \\ A_t k_t^\alpha &\approx A{k^*}^\alpha + \alpha A{k^*}^{\alpha-1}(k_t - k^*) + {k^*}^\alpha(A_t - A) \\ &\approx A{k^*}^\alpha + \alpha A{k^*}^{\alpha-1}k^*\hat{k}_t + A{k^*}^\alpha\hat{A}_t \\ &= A{k^*}^\alpha(1 + z_t + \alpha\hat{k}_t) \end{aligned}$$

(since $\hat{A}_t = z_t$). Plug this into the resource constraint:

$$c^*(1+\hat{c}_t) + k^*(1+\hat{k}_{t+1}) \approx A{k^*}^\alpha(1+z_t+\alpha\hat{k}_t) + (1-\delta)k^*(1+\hat{k}_t)$$

Linearizing the stochastic basic model II

We can still use

$$c^* = A{k^*}^{\alpha} - \delta k^*$$

since

$$c^*(1+\hat{c}_t) + k^*(1+\hat{k}_{t+1}) \approx A{k^*}^{\alpha}(1+z_t+\alpha\hat{k}_t) + (1-\delta)k^*(1+\hat{k}_t)$$

is then:

$$c^*\hat{c}_t + k^*\hat{k}_{t+1} = \left[1 + \alpha A{k^*}^{\alpha-1} - \delta\right] k^*\hat{k}_t + \alpha A{k^*}^{\alpha} z_t$$

or

$$c^*\hat{c}_t + k^*\hat{k}_{t+1} = \frac{1}{\beta}k^*\hat{k}_t + \alpha A{k^*}^{\alpha} z_t \qquad (3)$$

Linearizing the stochastic basic model III

What changes for the Euler equation? Having expectations do not change the approximations that much for the marginal utility terms:

$$u'(c_t) \approx u'(c^*) \left[1 + \frac{c^* u''(c^*)}{u'(c^*)} \hat{c}_t\right] \quad (Same)$$

$$\begin{aligned}\beta(1-\delta)E_t u'(c_{t+1}) &\approx \beta(1-\delta)E_t \left[u'(c^*) + u''(c^*)(c_{t+1} - c^*)\right] \\ &= \beta(1-\delta)u'(c^*) \left[1 + \frac{c^* u''(c^*)}{u'(c^*)} E_t \hat{c}_{t+1}\right]\end{aligned}$$

Linearizing the stochastic basic model IV

For the term involving A_{t+1} we need to remember that this is a variable too:

$$\alpha E_t \left[A_{t+1} k_{t+1}^{\alpha-1} \beta u'(c_{t+1}) \right]$$
$$\approx \alpha A {k^*}^{\alpha-1} \beta u'(c^*) + \alpha(\alpha-1) A {k^*}^{\alpha-2} \beta u'(c^*)(k_{t+1} - k^*) + \alpha A {k^*}^{\alpha-1} \beta u''(c^*) E_t(c_{t+1} - c^*)$$
$$+ \alpha {k^*}^{\alpha-1} E_t(A_{t+1} - A)$$
$$\approx \alpha A {k^*}^{\alpha-1} \beta u'(c^*) + \alpha(\alpha-1) A {k^*}^{\alpha-2} \beta u'(c^*) k^* \hat{k}_{t+1} + \alpha A {k^*}^{\alpha-1} \beta u''(c^*) c^* \hat{c}_{t+1}$$
$$+ \alpha A {k^*}^{\alpha-1} E_t(z_{t+1})$$
$$= \alpha A {k^*}^{\alpha-1} \beta u'(c^*) \left[1 - (1-\alpha)\hat{k}_{t+1} + E_t z_{t+1} + \frac{c^* u''(c^*)}{u'(c^*)} E_t \hat{c}_{t+1} \right]$$

Linearizing the stochastic basic model V

By once more defining

$$\theta = -\frac{c^* u''(c^*)}{u'(c^*)}$$

we can repeat the simplifications from before to get:

$$\hat{c}_t = E_t \hat{c}_{t+1} - \beta \frac{1}{\theta} \alpha A k^{*\,\alpha-1} \left[E_t z_{t+1} - (1-\alpha) \hat{k}_{t+1} \right] \tag{4}$$

Linearizing the stochastic basic model VI

End result? Two linearized conditions:

$$c^* \hat{c}_t + k^* \hat{k}_{t+1} = \frac{1}{\beta} k^* \hat{k}_t + \alpha A {k^*}^{\alpha} z_t$$

$$\hat{c}_t = E_t \hat{c}_{t+1} - \beta \frac{1}{\theta} \alpha A {k^*}^{\alpha - 1} \left[E_t z_{t+1} - (1-\alpha) \hat{k}_{t+1} \right]$$

as well as the definition of z_t:

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Linearizing the stochastic basic model VII

These three linear equations can be used to solve for the entire path of consumption and capital. But now we have *stochastic* difference equations which add another layer of difficulties. Most often: Let a computer do the job.

Using software to help us

We will come back to how we use a specific software for helping us solve for the optimal path. For now just take the solution for granted:

$$\textit{Linear equations} \;\Rightarrow\; \textit{Computer} \;\Rightarrow\; \textit{Solution}$$

What can we use the solution for?

Impulse-response functions

Assume that we start out in steady state, i.e. $\hat{c}_t = \hat{k}_t = z_t = 0$. Then there is a shock to technology: $\varepsilon_t = \Delta > 0$. How will consumption and capital respond? A graph that illustrates the response (assuming no other shocks in the future) is often referred to as an **impulse-response function**.

Impulse-response functions II

First look at our model to gain some intuition:

$$c^*\hat{c}_t + k^*\hat{k}_{t+1} = \frac{1}{\beta}k^*\hat{k}_t + \alpha A {k^*}^{\alpha} z_t$$

$$\hat{c}_t = E_t\hat{c}_{t+1} - \beta\frac{1}{\theta}\alpha A {k^*}^{\alpha-1}\left[E_t z_{t+1} - (1-\alpha)\hat{k}_{t+1}\right]$$

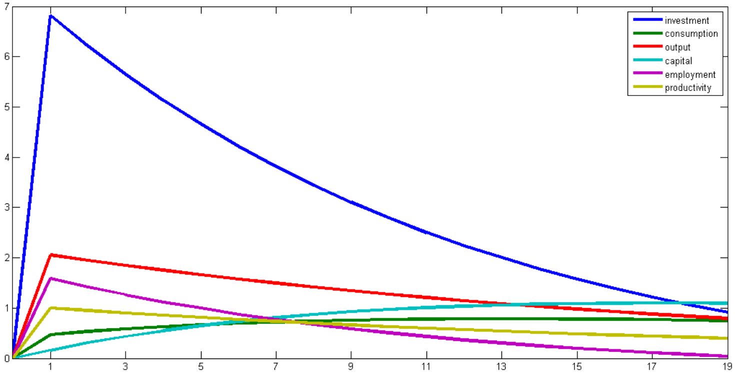
A shock to z_t has two effects:

- It increases available goods today
- It increases the rate of return on capital (provided that the shock is persistent – i.e. $\rho > 0$)

Let us look at the impulse-response for a model with $\beta = 0.99$, $\delta = 0.01$, $\alpha = 1/3$, $\theta = 1$, $A = 1$ and $\rho = 0.5$.

Impulse-response functions III

Solid: Consumption. Dashed: Capital. Effect of a one percentage point shock to ε_t. Since $\rho = 0.5$, z_t is quite persistent.

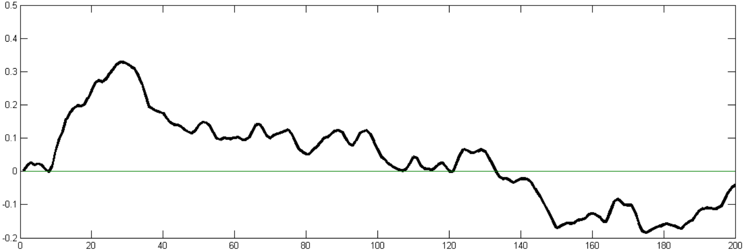


Impulse-response functions IV

Impulse-response functions is a common way to illustrate the dynamics of a model. So you should get used to seeing such plots!

Simulation

To generate artificial business cycle data, we draw realizations for $\{\varepsilon_t\}_{t=0}^{T}$, and solve for z_t, $\hat{c}_t$ and $\hat{k}_{t+1}$ (letting $\hat{k}_0 = 0$ be the initial condition). One realization of $\hat{c}_t$ is plotted here:



Simulation II

We see that even the basic model we've worked with so far, appended with a slightly persistent technology shock, manages to produce *data* that at least looks a bit like normal business cycles. But making cycles that look a bit like actual data is not enough: The BC facts reviewed in Lecture 9 give one dimension a good model should match.

Next steps?

Next steps involve

- Adding the labor supply dimension
- Then combining it all in a full RBC model