Labor supply in RBC models + calibration

Lecture 14, ECON 4310

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Summary from last lecture

Last lecture (#11) we went through

- What a solution is for our basic model
- How to linearize the deterministic model
- How to linearize the stochastic model

Summary from last lecture II

We saw in the deterministic case that the conditions describing optimum:

$$u'(c_t) = \beta[1 - \delta + \alpha A k_{t+1}^{\alpha-1}]u'(c_{t+1})$$
$$c_t + k_{t+1} = A k_t^{\alpha} + (1-\delta)k_t$$

could be linearized around steady state as:

$$\hat{c}_t = \hat{c}_{t+1} + \beta \frac{1}{\theta}(1-\alpha)\alpha A {k^*}^{\alpha-1}\hat{k}_{t+1}$$
$$c^* \hat{c}_t = \frac{1}{\beta} k^* \hat{k}_t - k^* \hat{k}_{t+1}$$

Summary from last lecture III

The conditions can be used to find

$$\hat{k}_{t+1} = a_2 \hat{k}_t$$

and

$$\hat{c}_t = \frac{k^*}{c^*}(\beta - a_2)\hat{k}_t$$

which is our *solution* to the model for an initial value of $\hat{k}_0$. The solution for the stochastic case is more complicated because of expectations, but very similar.

Today's lecture

- Introducing labor supply in the basic model
  - The *intratemporal* optimality condition
  - Frisch elasticity and IES for labor supply
- Labor lotteries
- The concept of calibration

Labor supply in the basic model

So far in the course we have considered models where a representative agent (or a social planner) maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

with some fixed amount of labor available for production. Now we consider the more general case where we maximze

$$\sum_{t=0}^{\infty} \beta^t U(c_t, h_t)$$

with h_t measuring hours worked, making $1 - h_t$ the hours of leisure.

Labor supply in the basic model II

In RBC models we will see that the labor supply response to changes in wages (driven by productivity shocks) is an important propagation mechanism.

Labor supply in the basic model III

To understand the basics, take one step back, and consider only a simple two-period model of labor supply, where we assume that utility is separable in consumption and labor supply:

$$\max_{\{c_0, c_1, h_0, h_1, a_1\}} u(c_0) - v(h_0) + \beta[u(c_1) - v(h_1)]$$

$$s.t.$$

$$c_0 + a_1 = w_0 h_0 + (1 + r_0) a_0$$

$$c_1 = w_1 h_1 + (1 + r_1) a_1$$

for a_0 given.

Labor supply in the basic model IV

This problem has the following first order conditions (letting λ_0 and λ_1 be the Lagrange multipliers)

$$u'(c_0) = \lambda_0 \quad (1)$$

$$\beta u'(c_1) = \lambda_1 \quad (2)$$

$$v'(h_0) = \lambda_0 w_0 \quad (3)$$

$$\beta v'(h_1) = \lambda_1 w_1 \quad (4)$$

$$\lambda_0 = \lambda_1 (1 + r_1) \quad (5)$$

Labor supply in the basic model V

- As before, combine (1), (2) and (3) to find the Euler equation:

$$u'(c_0) = \beta(1 + r_{t+1})u'(c_1)$$

  We refer to the Euler equation as the *intertemporal optimality condition*.

- Then to learn more about labor supply, combine (1) and (3) to find:

$$\frac{v'(h_0)}{u'(c_0)} = w_0$$

  This is a standard MRS = relative price condition. The LHS measures the utility loss (in terms of c_0) of one extra hour of work. The RHS gives the gain (in terms of c_0) from taking this hour of leisure. We refer to this as the *intratemporal optimality condition*.

- A similar condition holds of course for the last period:

$$\frac{v'(h_1)}{u'(c_1)} = w_1$$

Labor supply in the basic model VI

Notice that you can combine the Euler equation with the intratemporal optimality conditions to find:

$$\frac{v'(h_0)}{w_0} = \beta(1 + r_1)\frac{v'(h_1)}{w_1}$$

or:

$$\frac{\beta v'(h_1)}{v'(h_0)} = \frac{w_1}{(1 + r_1)w_0}$$

which we can refer to as the *intertemporal* labor supply condition. It is illustrating that we also face a choice along the intertemporal dimension when we choose labor supply.

Labor supply in the basic model VII

OK. Summary? We have one Euler equation and two intratemporal conditions:

$$u'(c_0) = \beta(1 + r_1)u'(c_1)$$
$$v'(h_0) = u'(c_0)w_0$$
$$v'(h_1) = u'(c_1)w_1$$

These three equations, together with the resource constraints:

$$c_0 + a_1 = w_0 h_0 + (1 + r_0)a_0$$
$$c_1 = w_1 h_1 + (1 + r_1)a_1$$

will determine the five endogenous variables c_0, c_1, h_0, h_1 and a_1.

Labor supply in the basic model VII

Assume that

$$u(c) - v(h) = \log c - \phi \frac{h^{1+\theta}}{1+\theta}$$

The Euler equation and the intratemporal conditions are in this case given by:

$$c_1 = \beta(1 + r_1)c_0$$
$$\phi h_0^\theta = \frac{w_0}{c_0}$$
$$\phi h_1^\theta = \frac{w_1}{c_1}$$

Labor supply in the basic model VIII

As we have seen before when utility of consumption is a log-function, we can combine the Euler equation with the resource constraints to find

$$c_0 = \frac{1}{1+\beta}\left[w_0 h_0 + \frac{w_1 h_1}{1+r_1}\right]$$

This solution for c_0, together with

$$\phi h_0^\theta = \frac{w_0}{c_0}$$
$$\phi h_1^\theta = \frac{w_1}{\beta(1+r_1)c_0}$$

are the conditions for optimum.

Labor supply in the basic model IX

Combining the intratemporal conditions we find

$$\left(\frac{h_1}{h_0}\right)^{\theta} = \frac{w_1}{\beta(1+r_1)w_0}$$

or

$$h_1 = \left(\frac{w_1}{\beta(1+r_1)w_0}\right)^{\frac{1}{\theta}} h_0$$

Labor supply in the basic model X

Then solve for h_0 by using the expressions for c_0 and h_1:

$$\begin{aligned}
\phi h_0^\theta &= \frac{w_0}{c_0} \\
\Rightarrow \phi h_0^\theta \left[w_0 h_0 + \frac{w_1 h_1}{1+r_1} \right] &= w_0(1+\beta) \\
\Rightarrow \phi h_0^\theta \left[w_0 h_0 + \frac{w_1}{1+r_1} \left(\frac{w_1}{\beta(1+r_1)w_0} \right)^{\frac{1}{\theta}} h_0 \right] &= w_0(1+\beta) \\
\Rightarrow \phi h_0^\theta \left[h_0 + \frac{w_1}{(1+r_1)w_0} \left(\frac{w_1}{\beta(1+r_1)w_0} \right)^{\frac{1}{\theta}} h_0 \right] &= (1+\beta) \\
\Rightarrow \phi h_0^{1+\theta} \left[1 + \left(\frac{w_1}{(1+r_1)w_0} \right)^{1+\frac{1}{\theta}} \beta^{-\frac{1}{\theta}} \right] &= (1+\beta)
\end{aligned}$$

Labor supply in the basic model XI

What is there to learn from this equation?

$$\phi h_0^{1+\theta}\left[1+\left(\frac{w_1}{(1+r_1)w_0}\right)^{1+\frac{1}{\theta}}\beta^{-\frac{1}{\theta}}\right]=(1+\beta)$$

- h_0 is increasing in w_0
- But it is also decreasing in w_1 (intertemporal substitution)
- An increase in w_0 and w_1 of the same relative size will not affect labor supply!
- So you get the result that if only w_0 goes up, then h_0 is also increased. But if w_0 and w_1 go up with w_1/w_0 constant, h_0 is unchanged. And if w_1 goes up, h_0 goes down.

[These conclusions are of course dependent on the utility function you use, but they illustrate general tendencies]

Two important elasticities

There are two important elasticities we need to care about:

1. **Frisch elasticity**: The elasticity of labor supply with respect to the wage, keeping marginal utility of wealth constant. *Measures the substitution effect*
2. **Intertemporal elasticity of substitution (IES) for labor supply**: The elasticity of relative labor supply *across* periods with respect to the present value of wage growth

Frisch elasticity

How to find the Frisch elasticity? Use the intratemporal optimality condition.

$$\frac{v'(h_t)}{u'(c_t)} = w_t$$

for $t = 0, 1$. For a given marginal utility of consumption, this defines an implicit function $h_t = q(w_t)$. Let us differentiate with respect to w_t:

$$\frac{v''(q(w_t))q'(w_t)}{u'(c_t)} = 1$$

Frisch elasticity II

Then we multiply by $v'(q(w_t))/q(w_t)$:

$$\frac{v'(q(w_t))}{q(w_t)} \frac{v''(q(w_t))q'(w_t)}{u'(c_t)} = \frac{v'(q(w_t))}{q(w_t)}$$

Divide both sides by $v''(q(w_t))$ and re-arrange the terms on the left to get

$$El_{w_t} h_t = El_{w_t} q(w_t) = \frac{w_t}{q(w_t)} q'(w_t) = \frac{v'(h_t)}{h_t v''(h_t)}$$

This is the Frisch elasticity of labor supply.

Frisch elasticity III

Continue using our last choice for $v(h)$:

$$v(h_t) = \phi \frac{h_t^{1+\theta}}{1+\theta}$$

With this, $v'(h) = \phi h_t^{\theta}$ and $v''(h) = \theta \phi h^{\theta - 1}$, implying:

$$El_{w_t} h_t = \frac{\phi h_t^{\theta}}{h_t \theta \phi h_t^{\theta - 1}} = \frac{1}{\theta}$$

i.e. a constant Frisch elasticity at $1/\theta$.

IES for labor supply

What about the IES for labor supply? Keep the particular choice of $v(h)$. To find this elasticity, we use the *inter*temporal optimality condition for labor:

$$\frac{\beta v'(h_1)}{v'(h_0)} = \frac{w_1}{(1+r_1)w_0}$$

which now becomes

$$\beta \left(\frac{h_1}{h_0}\right)^{\theta} = \frac{w_1}{(1+r_1)w_0} = \tilde{W}_0$$

where $\tilde{W}_1$ denotes the present value of wage growth.

IES for labor supply II

The IES for labor supply is the elasticity of h_1/h_0 with respect to $\tilde{W}_0$. To find it, we can either find derivatives etc. like for the Frisch case, or simply use that:

$$El_x y = \frac{d \log y}{d \log x}$$

Taking logs of the intertemporal optimality condition for labor we get:

$$\log \beta + \theta \log(\frac{h_1}{h_0}) = \log \tilde{W}_0$$

Hence:

$$El_{\tilde{W}_0} \frac{h_1}{h_0} = \frac{1}{\theta}$$

In this case the IES for labor supply equals the Frisch elasticity.

Using the elasticities

- The higher the Frisch elasticity, the more willing are you to work if the wage increases
- The higher the IES for labor supply, the more willing are you to shift the *path* of labor supply in response to temporary changes in the wage

Using the elasticities II

With $v(h) = \phi \frac{h^{1+\theta}}{1+\theta}$:

- Empirical estimates of the Frisch elasticity are often in the range of 0.5, implying $\theta = 2$
- In contrast, maximum volatility in hours is obtained by setting $\theta = 0$ (since then the Frisch elasticity $\to \infty$). This would make

$$v(h) = \phi h$$

  i.e. linear in hours.

Using the elasticities III

- Since we want to choose values for our structural parameters that are consistent with micro evidence, we should also set θ close to 2 in an RBC model.
- But values of θ around 2 are often producing too little volatility in labor supply in RBC models!
- To get more volatile labor supply, one would rather be somewhere closer to $\theta = 0$, in which case $v(h)$ is linear in h and we get maximium volatility.
- This is a problem

But we know that (e.g. as shown in Kydland and Prescott, 1990) fluctuations in labor supply seems to be driven primarily by changes in the *extensive* margin – not so much by the intensive. Can we change our model to account for this?

Labor lotteries

This is the motivation for models of *indivisible* labor combined with labor lotteries (see Hansen (1984) and Rogerson (1988)).

- In the simple model the agent could choose h to be anywhere between zero and one
- With indivisible labor, we will require $h = \{0, 1\}$, i.e. working becomes a 'yes/no' choice
- Labor lotteries (Rogerson, 1988) offers an elegant way of introducing this mechanism

Labor lotteries II

Consider the following setting:

- There exists a continuum of households on the unit interval, each with a utility function $\sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)]$
- Hours worked must by each agent is either 0 or 1
- All agents agree to join in a 'labor lottery': With probability ξ_t they will have to work, and with probability $1 - \xi_t$ they will be unemployed. But no matter if they work or not, all will recieve the same income (and therefore consumption).
- ξ_t is then chosen by the group or a social planner to maximize welfare
- With a continuum of agents, ξ_t can be interpreted as the *share* of agents that must work

Labor lotteries III

Since all agents are the same, we maximize welfare by maximizing

$$E\left\{\sum_{t=0}^{\infty}\beta^t[u(c_t)-v(h_t)]\right\} = E\left\{\sum_{t=0}^{\infty}\beta^t[u(c_t)-v(h_t)]|\text{Work}\right\} + E\left\{\sum_{t=0}^{\infty}\beta^t[u(c_t)-v(h_t)]|\text{Not work}\right\}$$
$$= \sum_{t=0}^{\infty}\xi_t\beta^t[u(c_t)-v(1)] + \sum_{t=0}^{\infty}(1-\xi_t)\beta^t[u(c_t)-v(0)]$$
$$= \sum_{t=0}^{\infty}\beta^t[u(c_t)-\xi_t v(1)-(1-\xi_t)v(0)]$$
$$= \sum_{t=0}^{\infty}\beta^t[u(c_t)-\xi_t[v(1)-v(0)]-v(0)]$$

Let us define $D = v(1) - v(0)$ and ignore the last $v(0)$ term (since a constant is not relevant for maximizing a function). The objective function we are left with is

$$\sum_{t=0}^{\infty}\beta^t[u(c_t)-D\xi_t]$$

Labor lotteries IV

But this is like magic! We started out with an economy where every agent was identical, such that the social planner problem would be to maximize

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - v(h_t)]$$

Introducing labor lotteries instead, gives us:

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - D\xi_t]$$

where ξ_t can be interpreted as our new 'labor supply' since total labor supply n_t must equal ξ_t. This latter utility function is *linear* in labor supply, which gives us hope that it will also give larger labor supply responses when shocks are hitting the economy.

Labor lotteries V

- Recall, if we have

$$v(h) = \phi \frac{h^{1+\theta}}{1+\theta}$$

  then $\frac{1}{\theta}$ is the Frisch elasticity.
- We can set $\theta = 2$ to have micro elasticities that are plausible
- For the model with labor lotteries, the value of θ only affects D, since:

$$D = v(1) - v(0) = \frac{\phi}{1+\theta}$$

  so it does not affect the substitution effects.
- Since the labor lotteries model gives us a model *as if* utility was linear, we get a **macro** Frisch elasticity equal to infinity, no matter what we set the micro elasticity to be!
- So there is a difference between micro and macro elasticities

Labor lotteries VI

Intuition for the possible difference between micro and macro elasticities:

- For the micro elasticity, we look at the effect on hours worked from a marginal change in the wage. When hours are changing, your disutility of labor change as well, dampening the impact
- For a macro elasticity, we only look at the effect on aggregate hours worked when the wage level changes. If all labor is indivisible, all changes in ours are due to people going from unemployment to employment. Their disutility of work is constant since work is a zero-one choice. So there is no dampening effect from changes in disutility of labor.

Labor lotteries VII

RBC models therefore often assume utility functions where utility is linear in labor supply, using a labor lottery argument as fundament.

Basic model with labor lottery

Our basic model combined with a labor lottery assumption gives then the following social planner's problem:

$$\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [u(c_t) - Dn_t]$$

$$s.t.$$

$$c_t + k_{t+1} = Ak_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t$$

$$c_t \geq 0$$

$$k_{t+1} \geq 0$$

$$0 \leq n_t \leq 1$$

with $k_t > 0$ given. We continue to 'ignore' the conditions of c, k and n, since we will find an interior solution.

Basic model with labor lottery II

Form the Lagrangian as before (λ_t being the Lagrange multiplier), and find the first-order conditions.

- With respect to c_t:

$$\beta^t u'(c_t) = \lambda_t \tag{6}$$

- With respect to n_t:

$$D = \lambda_t A(1-\alpha) k_t^{\alpha} n_t^{-\alpha} \tag{7}$$

- With respect to k_{t+1}:

$$\lambda_t = \lambda_{t+1}[A\alpha k_{t+1}^{\alpha-1} n_t^{1-\alpha} + 1 - \delta] \tag{8}$$

Basic model with labor lottery III

- As before, combine (1) and (3) to find the Euler equation:

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$$

  where $r_{t+1} = A\alpha k_{t+1}^{\alpha-1} n_t^{1-\alpha} - \delta$.

- Combine (1) and (2) to find the intratemporal optimality condition:

$$\frac{D}{u'(c_t)} = w_t$$

  where $w_t = A(1-\alpha)k_t^{\alpha} n_t^{-\alpha}$

Basic model with labor lottery IV

With n fixed (before today), optimum required the following conditions to be satisfied:

- The Euler equation
- The resource constraint

Introducing labor supply and making n be set optimally adds one extra restriction:

- The intratemporal optimality condition

Basic model with labor lottery V

Next steps? Like in Lecture 11:

- Characterize steady state
- Linearize conditions around steady state
- Solve the set of linearized equations
- Plot impulse-response functions, simulate, calculate moments etc.

We can save this to next lecture.

Calibration

One thing we will not save to next lecture is: **How should we choose values for the structural parameters in an RBC model?** What is most frequently applied is called *calibration*.

Calibration II

Take the basic model with labor lottery. Assume that the utility function is

$$u(c) = \log c$$

Ignoring productivity, the model as four structural parameters:

- Discount factor β
- Deprecitation rate δ
- Cobb-Douglas parameter α
- Disutility of labor supply D

To calibrate the model we must find four *moments* (usually averages) we want our model to match. By this we mean that the *steady state* properties of the model should match the data.

Calibration III

A standard set of moments to match are:

- Average capital share of income
- Average investment to capital ratio
- Average long-term real interest rate
- Average share of available hours spent on work

Let us see how we can use each of these moments to calibrate our model.

Calibration IV

Start with the average capital share. Say that we have observed an average US capital share of 1/3 over the last 50 years. To use this fact, let us calculate what the capital share in our model is:

$$\frac{r_t k_t}{y_t} = \frac{\alpha A k_t^{\alpha-1} n_t^{1-\alpha} k_t}{A k_t^{\alpha} n_t^{1-\alpha}} = \alpha$$

So if we set $\alpha = 1/3$, we ensure that the model implies a realistic capital share.

Calibration V

Then take the average investment to capital ratio. Usually we only observe $\frac{I}{Y}$ and $\frac{K}{Y}$. Say that we've calculated an average investment to output share of 0.25 and capital to output share of 10. To use this fact, let us look at the law of motion for capital

$$k_{t+1} = (1 - \delta)k_t + i_t$$

Divide by output and use that k_t/y_t is constant in steady state. That gives us:

$$\delta = \frac{i}{k} = \frac{i}{y}\left(\frac{k}{y}\right)^{-1}$$

So if we set $\delta = 0.025$, we ensure that the model implies a realistic investment to capital ratio in steady state.

Calibration VI

Then there is the long-term interest rate. If our model is quarterly, it could be that 1% real interest rate is realistic. The Euler equation in steady state (constant consumption) gives us:

$$1 = \beta(1 + r)$$

or

$$\beta = \frac{1}{1 + r}$$

So if we set $\beta = 1/1.01 \approx 0.99$, we ensure that the model implies a realistic real interest rate in steady state.

Calibration VII

Finally: the share of hours available that is spent on work. Maybe $n = 1/3$ is realistic. Use the intratemporal optimality condition in steady state:

$$\frac{D}{u'(c)} = w$$

When $u(c) = \log c$ this can be written as

$$D = \frac{w}{c} = \frac{1}{n}\frac{wn}{c} = \frac{1}{n}\frac{(1-\alpha)y}{c} = \frac{1}{n}\frac{1-\alpha}{c/y} = \frac{1}{n}\frac{1-\alpha}{1-i/y}$$

With $n = 1/3$, $\alpha = 1/3$ and $i/y = 0.25$, this gives

$$D = \frac{1}{1/3}\frac{1-1/3}{1-0.25} = \frac{8}{3}$$

So if we set $D = 8/3$, we ensure that the model implies a realistic share of hours spent on work.

Calibration VIII

Summary? If we want to ensure a capital share equal to $1/3$, an investment to capital ratio of 2.5%, a real interest rate of 1% and $n = 1/3$ in our model we just choose:

- $\alpha = 1/3$
- $\delta = 0.025$
- $\beta = 1/1.01$
- $D = 8/3$

Calibration IX

- Calibrating the model in this way ensures that the model has reasonable *long-run* properties.
- So it is not impressing that our RBC model manages to replicate these facts
- The interesting question is: How well will a simple model calibrated to match long-run facts do when it comes to explain business cycles?

Some more on calibration of labor supply

What does $D = 8/3$ imply for the parameters in $v(h)$? We keep on assuming

$$v(h) = \phi \frac{h^{1+\theta}}{1+\theta}$$

so that $D = \phi/(1+\theta)$. This shows that if we want $\theta = 2$ (to be consistent with micro data), we need $\phi = 8$.

Some more on calibration of labor supply II

Then imagine that we were back to the model with *divisible* labor. In that model the intratemporal optimality condition in steady state is

$$\frac{v'(h)}{u'(c)} = w$$

or with $v(h)$ as specified and log utility:

$$\phi h^{\theta} = \frac{w}{c}$$

Doing the same transformations on the RHS as earlier, we get

$$\phi h^{\theta} = \frac{1}{h} \frac{1-\alpha}{1-i/y}$$

which gives

$$\phi = \frac{1}{h^{1+\theta}} \frac{1-\alpha}{1-i/y}$$

If $\theta = 2$ (and the remaining calibration is as before), we have $\phi = 24$.

Some more on calibration of labor supply III

- So we could of course also calibrate a model with divisible labor to obtain $h = 1/3$ in steady state. The effect is an implicit selection of a much larger value of ϕ.
- But that does not change the main difference between the labor lottery and divisible labor models: The difference in substitution effects!

Three things you MUST remember from today

1. What is the Frisch elasticity?
2. Why do we use the labor lotteries model?
3. How do we choose values for the structural parameters in an RBC model?