How much should the nation save?

Econ 4310 Lecture 2

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Outline

- Golden Rule
- Social planner's choice
- Market solution
- Bringing back natural growth

Slide set covers first two, lecture may go further

The Golden Rule of Accumulation

Consumption per efficiency unit of labor in steady state is:

$$c = f(k) - \gamma k \tag{1}$$

First order condition for maximum is $f'(k) - \gamma = 0$. Golden rule level of k, k^{**} is determined by

$$f'(k^{**}) = \gamma \tag{2}$$

$$r^{**} = \gamma$$

Interest rate equal to natural growth rate
Savings rate required to reach k^{**}:

$$s^{**} = \gamma k^{**}/f(k^{**}) = r^{**}k^{**}/f(k^{**})$$

Along the Golden rule path the savings rate equals the income share of capital.
If s is increased beyond s^{**}, consumption is reduced both now and in all future!

The Ramsey model, simple version

Social planner maximizing utility over all future generations given

- Zero population growth
- zero productivity growth

Welfare function

$$U_0 = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + ... = \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad (1)$$

$0 < \beta < 1$ discount factor (subjective)

$\beta = \frac{1}{1+\rho}$ $\rho =$ discount rate, degree of impatience

$u(C)$ period utility $u' > 0$ $u'' < 0,$ $u'(0) = \infty$

The accumulation equation

$$y_t = f(k_t)$$

$$k_{t+1} = k_t + i_t$$

$$k_{t+1} = k_t + f(k_t) - c_t \Leftrightarrow c_t = f(k_t) + k_t - k_{t+1} \qquad (2)$$

- y_t output
- k_t capital
- c_t consumption

The maximization problem

$$\max U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

given

$$c_t = f(k_t) + k_t - k_{t+1} \text{ and } k_t \geq 0,\ k_0 = \bar{k_0}$$

Insert for c_t from accumulation equation in utility function.

Solution

Utility function after insertions is :

$$U_0 = \ldots + \beta^t u(f(k_t) + k_t - k_{t+1}) + \beta^{t+1} u(f(k_{t+1}) + k_{t+1} - k_{t+2}) + \ldots$$

$k_t + 1$ appears only in terms t and $t + 1$
Necessary condition for maximum are:

$$\frac{\partial U_0}{\partial k_{t+1}} = -\beta^t u'(c_t) + \beta^{t+1} u'(c_{t+1})(1 + f'(k_{t+1})) = 0$$

or

$$u'(c_t) = \beta u'(c_{t+1})(1 + f'(k_{t+1})) \text{ for } t = 0, 1, 2, \ldots, \infty \qquad (3)$$

The Consumption Euler-equation

$$u'(c_t) = \beta u'(c_{t+1})(1 + f'(k_{t+1})) \text{ for } t = 0, 1, 2, \ldots, \infty \tag{4}$$

- A marginal transfer of resources between periods t and $t+1$ should not increase total welfare
- A marginal transfer of resources between any two periods should not increase total welfare

$$\frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{1}{1 + f'(k_{t+1})} \text{ for } t = 0, 1, 2, \ldots, \infty \tag{5}$$

- Marginal rate of substitution should equal marginal rate of transformation

Problem not solved yet

Two difference equations:

$$k_{t+1} = f(k_t) + k_t - c_t \quad (6)$$

$$u'(c_{t+1}) = [\beta(1 + f'(k_{t+1}))]^{-1} u'(c_t) \quad (7)$$

Non-linear system in two unknown time series. One initial condition (k_0 given)

- Exclude paths where $k_t \to 0$ when $t \to \infty$ (resource constraint).
- Choose the path with highest utility among the remaining

- Closed form solutions only in special cases. Simulation usually necessary.

The stationary state

Definition:

$$k_{t+1} = k_t = k^* \text{ and } c_{t+1} = c_t = c^*$$

Insert in difference equations (**??**) and (**??**):

$$\begin{aligned} k^* &= f(k^*) + k^* - c^* \\ u'(c^*) &= [\beta(1 + f'(k^*))]^{-1} u'(c^*) \end{aligned}$$

Solution:

$$f'(k_*) = (1-\beta)/\beta = \rho \quad (8)$$

$$c^* = f(k^*) \quad (9)$$

- Subjective discount rate determines capital intensity.
  item Capital intensity determines output.
- No savings, consumption = output

Comparison to golden rule

Remember: No population growth, no productivity growth

- Golden rule: $f'(k^*) = 0$
- Ramsey rule: $f'(k^*) = \rho > 0$
- Social planner chooses less capital

Graphical solution

Phase diagram will be drawn to show:

- Social planner will choose a path that leads to the stationary state
- If$k_0 < k_*$,both consumption increases gradually as the economy moves towards the steady state.

Neoclassical growth model

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Consumers

$$\max U_0 = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + ... = \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad (1)$$

s.t.

$$c_t = (1 + r_t)a_t + w_t - a_{t+1} \qquad t = 0, 1, 2, \ldots \qquad (2)$$

$$\lim_{t \to \infty} a_t R_{t-1}^{-1} \geq 0 \qquad R_t = \prod_{s=0}^{t} (1 + r_s) \qquad (3)$$

a_0 given

a_t = net assets carried over from period $t - 1$, period t, r_t = interest rate paid on these

Budget constraint, finite horizon

Constant interest rate, two periods:

$$\begin{aligned} a_1 &= a_0(1+r) + w_0 - c_0 \\ a_2 &= (1+r)a_1 + w_1 - c_1 \\ a_2 &= (1+r)^2 a_0 + (1+r)[w_0 - c_0] + [w_1 - c_1] \end{aligned}$$

Budget constraint: $a_2 \geq 0$. Equivalent formulation:

$$a_2(1+r)^{-1} = a_0(1+r) + [w_1 - c_1] + (1+r)^{-1}[w_2 - c_2] \geq 0$$

Budget constraint with infinite horizon

$\lim_{t\to\infty} a_t \geq 0$ Debt must be paid back!

$\lim_{t\to\infty}(1+r)^{-(t-1)}a_t \geq 0$ Debt can be rolled over forever!

Budget constraint Continued

$$\lim_{t \to +\infty} (1+r)^{-(t-1)} a_t \geq 0$$

Beyond some point $\frac{a_t}{a_{t-1}} \geq r$

- Debt cannot grow faster than the interest rate
- Some interest has to be paid from current income
- Consumption has to be below wage income

Present value version

Assets accumulated at the end of period $t-1$:

$$a_t = a_0(1+r)^t + \sum_{j=0}^{t-1}(w_j - c_j)(1+r)^{t-1-j}$$

Present value of a_t is:

$$a_(1+r)^{-(t-1)} = a_0(1+r) + \sum_{j=0}^{t-1}(w_j - c_j)(1+r)^{-j}$$

Taking limits on both sides, we get

$$\lim_{t\to\infty}(1+r)^{-(t-1)}a_t = a_0(1+r) + \sum_{j=0}^{\infty}(w_j - c_j)(1+r)^{-j} \geq 0$$

or

$$\sum_{j=0}^{\infty} c_j(1+r)^{-j} \leq a_0(1+r)\sum_{j=0}^{\infty} w_j(1+r)^{-j} \tag{4}$$

First-order conditions

Use budget equation to substitute for c in utility:

$$U_0 = \dots \beta^t u((1+r_t)a_t + w_t - a_{t+1}) + \beta^{t+1} u((1+r_{t+1})a_{t+1} + w_{t+1} - a_{t+2}) + \dots$$

Differentiate with respect to a_{t+1}

$$\frac{\partial U_0}{\partial a_{t+1}} = \beta^t u'(c_t)(-1) + \beta^{t+1} u'(c_{t+1})(1 + r_{t+1}) =$$

Consumption Euler equation

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1}) \qquad t = 1, 2, \dots \tag{5}$$

Terminal condition

Budget constraint should be satisfied with equality

$$\lim_{t\to\infty} a_t R_{t-1}^{-1} = 0 \tag{6}$$

- Assets should not grow faster than interest rate
- A part of interest income should be spent

Demand functions

Solution to optimization problem has to satisfy

- First order conditions
- Period by period budget equations
- Present value budget constraint with =

$$C_t = D_t(W_0, r_1, r_2, r_3, \ldots) \tag{7}$$

- Consumption depends on total wealth, not on income each year
- The distribution of consumption on periods is independent of the distribution of income

Labor and capital

$$f'(k_t) = r_t \tag{8}$$

$$f(k_t) - k_t f'(k_t) = w_t \tag{9}$$

Equilibrium conditions

$$a_t = k_t \tag{10}$$

$$c_t = k_{t+1} = k_t + f(k_t) \tag{11}$$

$$k_t \geq 0 \quad t = 1, 2, 3 \ldots, \infty \tag{12}$$

Relation to planner's optimum

From combining Euler and $r = f'(k)$:

$$u'(c_t) = \beta u c_{t+1})(1 + f'(k_{t+1})) \tag{13}$$

Same difference equations, same stationary point, same phase diagram Paths whre $k \to 0$, violate the budget constraint Paths where $k \to \infty$ do not use the whole budget; as r goes to zero consumption stays below wage income forever