Neoclassical growth model

Econ 4310 Lecture 3

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Consumers

$$\max U_0 = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + ... = \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad (1)$$

s.t.

$$c_t = (1 + r_t)a_t + w_t - a_{t+1} \qquad t = 0, 1, 2, \dots \qquad (2)$$

$$\lim_{t\to\infty} a_t R_{t-1}^{-1} \geq 0 \qquad R_t = \prod_{s=0}^{t} (1 + r_s) \qquad (3)$$

$$a_0 \text{ given}$$

a_t = net assets carried over from period $t - 1$ to period t, r_t = interest rate paid on these

Budget constraint, finite horizon

Constant interest rate, two periods:

$$\begin{aligned} a_1 &= a_0(1+r) + w_0 - c_0 \\ a_2 &= (1+r)a_1 + w_1 - c_1 \\ a_2 &= (1+r)^2 a_0 + (1+r)[w_0 - c_0] + [w_1 - c_1] \end{aligned}$$

Budget constraint: $a_2 \geq 0$. Equivalent formulation:

$$a_2(1+r)^{-1} = a_0(1+r) + [w_1 - c_1] + (1+r)^{-1}[w_2 - c_2] \geq 0$$

Budget constraint with infinite horizon

$\lim_{t\to\infty} a_t \geq 0$ Debt must be paid back!

$\lim_{t\to\infty}(1+r)^{-(t-1)}a_t \geq 0$ Debt can be rolled over forever!Interest must be paid from earned income. No Ponzi-Game
Constant versus variable r:

$$R_{t-1} = (1+r)^{t-1} \text{ vs } R_{t-1} = (1+r_0)(1+r_1)\cdots(1+r_{-t-1}) = \prod_{s=0}^{t-1}(1+r_s)$$

Budget constraint continued

$$\text{NPG} \quad \lim_{t\to\infty}(1+r)^{-(t-1)}a_t \geq 0 \tag{4}$$

- If $a_t \geq 0$ for all $t \geq$ some t', NPG is always satisfied
- If $a_t < 0$ for all $t \geq$ some t', NPG can be satisfied only if the rate of increase in the debt is below the interest rate:

$$\frac{|a_t|}{|a_{t-1}|} < 1 + r \Leftrightarrow a_t > (1+r)a_{t-1} \tag{5}$$

- The last inequality requires that some interest is paid from current earnings, not by taking on new debt.
- Condition (5) is necessary, but not sufficient for (4) to hold when $a < 0$. The amount that is paid need to keep pace with size of debt.

More on constraint

- A sufficient condition is that a constant share of the interest payments are paid out of current earnings. However, when this share is below 100 per cent, the amount paid from current earnings goes to infinity as the deb increases.
- If there is no trend growth, NPG can only be satisfied if in the limit *all* interest is paid from current earnings.
- In the present model this limits household borrowing to w/r.
- NPG is always satisfied if a constant debt is rolled over forever:
- A constant debt requires that all interest is paid from current income or $c_t = w_t - rb$ where b is the size of the debt.

Present value version

Assets accumulated at the end of period $t-1$:

$$a_t = a_0(1+r)^t + \sum_{j=0}^{t-1}(w_j - c_j)(1+r)^{t-1-j}$$

Present value of a_t is:

$$a_(1+r)^{-(t-1)} = a_0(1+r) + \sum_{j=0}^{t-1}(w_j - c_j)(1+r)^{-j}$$

Taking limits on both sides, we get

$$\lim_{t\to\infty}(1+r)^{-(t-1)}a_t = a_0(1+r) + \sum_{j=0}^{\infty}(w_j - c_j)(1+r)^{-j} \geq 0$$

or

$$\sum_{j=0}^{\infty} c_j(1+r)^{-j} \leq a_0(1+r)\sum_{j=0}^{\infty} w_j(1+r)^{-j} \tag{6}$$

First-order conditions

Use budget equation to substitute for c in utility:

$$U_0 = \dots \beta^t u((1+r_t)a_t + w_t - a_{t+1}) + \beta^{t+1} u((1+r_{t+1})a_{t+1} + w_{t+1} - a_{t+2}) + \dots$$

Differentiate with respect to a_{t+1}

$$\frac{\partial U_0}{\partial a_{t+1}} = \beta^t u'(c_t)(-1) + \beta^{t+1} u'(c_{t+1})(1 + r_{t+1}) = 0$$

Consumption Euler equation

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1}) \qquad t = 1, 2, \dots \tag{7}$$

Terminal condition

Budget constraint should be satisfied with equality

$$\lim_{t \to \infty} a_t R_{t-1}^{-1} = 0 \tag{8}$$

- If positive, assets should not grow faster than interest rate
- A part of interest income should be spent

Demand functions

Solution to optimization problem has to satisfy

- First order conditions
- Period by period budget equations
- Present value budget constraint with =

Demand functions

$$C_t = D_t(W_0, r_1, r_2, r_3, \ldots) \tag{9}$$

- Consumption depends on total wealth, not on income each year
- The distribution of consumption on periods is independent of the distribution of income

Labor and capital

$$f'(k_t) = r_t \tag{10}$$

$$f(k_t) - k_t f'(k_t) = w_t \tag{11}$$

Equilibrium conditions

$$a_t = k_t \quad (12)$$

$$c_t + k_{t+1} = k_t + f(k_t) \quad (13)$$

$$k_t \geq 0 \;\; t = 1, 2, 3 \ldots, \infty \quad (14)$$

Relation to planner's optimum

From combining Euler and $r = f'(k)$:

$$u'(c_t) = \beta u'(c_{t+1})(1 + f'(k_{t+1})) \tag{15}$$

- Same difference equations, same stationary point, same phase diagram
- Paths where $k \to 0$, violate the budget constraint
- Paths where $k \to \infty$ do not use the whole budget; as r goes to zero consumption stays below wage income forever

Competitive markets, Pareto-efficiency and welfare optimum

Ramsey-model with trend growth

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New and noteworthy

Population growth and technology growth

$$A_t = A_0(1+g)^t, \qquad L_t = L_0(1+n)^t$$

- Natural growth rate $(1+\gamma) = (1+n)(1+g)$
- Utilitarian social planner

The Social planner's problem

$$\max U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t A_t) L_t \tag{1}$$

given

$$c_t = f(k_t) + k_t - (1+\gamma)k_{t+1}, \tag{2}$$

$$k_0 = \bar{k_0}, \quad k_t \geq 0, \quad c_t \geq 0$$

Insert for c_t in utility function and take derivatives to find first order condition.
c and k are measured per efficiency unit of labor

First order condition

$$\frac{\partial U}{\partial k_{t+1}} = - \beta^t [u'(c_t A_t) A_t L_t (1+\gamma)] + \beta^{t+1} [u'(c_{t+1} A_{t+1}) A_{t+1} L_{t+1} (1 + f' k_{t+1})] = 0 \qquad (3)$$

Divide through by $\beta^t A_t L_t (1+\gamma)$ and move one term to each side:

$$u'(c_t A_t) = \beta u'(c_{t+1} A_{t+1})(1 + f'(k_{t+1})) \qquad (4)$$

Comparison

Without natural growth

$$u'(c_t) = \beta u'(c_{t+1})(1 + f'(k_{t+1})) \tag{5}$$

With natural growth

$$u'(c_t A_t) = \beta u'(c_{t+1} A_{t+1})(1 + f'(k_{t+1})) \tag{6}$$

- Balanced growth possible only if $u'(c(1+\gamma))/u'(c)$ independent of c

Warning!

- There may be no maximum!
- Discounted utility may be infinite! Low discount rate, high productivity growth

CRRA-preferences

$$u(x) = \frac{1}{1-\theta} c^{1-\theta,} \quad \theta > 0$$

- Constant relative risk aversion
- Degree of relative risk aversion is θ
- Intertemporal elasticity of substitution is $\sigma = 1/\theta$
- Marginal utility is

$$u'(c) = c^{-\theta}$$

- High σ means indifference curves are less curved

First order condition with CRRA

General:

$$u'(c_t A_t) = \beta u'(c_{t+1} A_{t+1})(1 + f'(k_{t+1}))$$

With CRRA:

$$(c_t A_t)^{-\theta} = \beta (c_{t+1} A_{t+1})^{-\theta}(1 + f'(k_{t+1}))$$

$$\frac{(c_t A_t)^{-\theta}}{\beta (c_{t+1} A_{t+1})^{-\theta}} = 1 + f'(k_{t+1})$$

- MRS = MRT
- MRS depends only on ratio between consumption in t and $t+1$
- Confirms that elasticity of substitution is $1/\theta$

$$\frac{c_{t+1} A_{t+1}}{c_t A_t} = [\beta(1 + f'(k_{t+1}))]^{\sigma}$$

Consumption growth rates

$$\beta = 1/(1+\rho)$$

Per capita:

$$\frac{c_{t+1}A_{t+1}}{c_t A_t} = [\beta(1 + f'(k_{t+1}))]^\sigma$$

Per efficiency unit of labor:

$$\frac{c_{t+1}}{c_t} = [\beta(1 + f'(k_{t+1}))]^\sigma/(1+g)$$

Consumption per capita should be growing if $r_t = f'(k_{t+1}) > \rho$.

Difference equations

$$k_{t+1} = \frac{1}{1+\gamma}[k_t + f(k_t) - c_t] \tag{7}$$

$$c_{t+1} = c_t \left[\frac{1 + f'(k_{t+1})}{1+\rho}\right]^{\sigma} (1+g)^{-1} \tag{8}$$

Get stationarity conditions by inserting from

$$k_{t+1} = k_t = k* \text{ and } c_{t+1} = c_t = c*$$

Conditions for balanced growth

$$c^* = f(k^*) - [(1+g)(1+n) - 1]k_* = f(k^*) - \gamma k^* \tag{9}$$

$$\left(\frac{1 + f'(k^*)}{1+\rho}\right)^{\sigma} \frac{1}{1+g} = 1 \tag{10}$$

or:

$$(1 + f'(k^*) = (1+\rho)(1+g)^{1/\sigma} \tag{11}$$

Loglinearizing the steady state condition 10

$$(1+g)^{-1}(1+\rho)^{-\sigma}[(1+f'(k_*))]^{\sigma} = 1$$

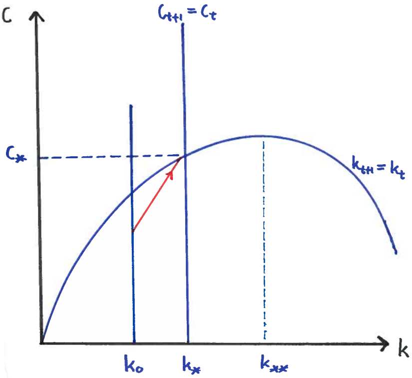
$$-\ln(1+g) - \sigma\ln(1+\rho) + \sigma\ln(1+f'(k_*)) = \ln 1 = 0$$

$$-g - \sigma\rho + \sigma f'(k_*) \approx 0$$

$$f'(k_*) \approx \rho + \frac{1}{\sigma}g \tag{12}$$

Short period, g, ρ and $f'(k)$ small numbers, $\ln(1+x) \approx x$

Convergence to balanced growth path



$$c = f(k) - \gamma k$$

Observations on the steady state

$$r^* = f'(k^*) \approx \rho + \frac{1}{\sigma} g$$

- k^* is independent of n
- k^* depends negatively on ρ
- k^* depends negatively on g and more so the lower is σ
- k_* depends positively on σ when $g > 0$

Real interest rate depends positively on g, ρ

Observations on the steady state

$$r^* = f'(k^*) \approx \rho + \frac{1}{\sigma} g$$

High$g$

- $\rightarrow$ Future generations benefit from higher productivity
- $\rightarrow$ Less reason to save
- $\rightarrow$ Less investment
- $\rightarrow$ Capital stock stops growing at a lower level

Low σ

- $\rightarrow$ Strong preference for smooth consumption path
- $\rightarrow$ More consumption and less saving to begin with
- $\rightarrow$ Capital intensity stops at a lower level.

Numerical example

σ	ρ	g	$f'(k_*)$
0.5	0.02	0.03	0.08
0.5	0.02	0.04	0.10
1.0	0.02	0.03	0.05
1.0	0.02	0.02	0.04

$$f'(k_*) \approx \rho + \frac{1}{\sigma} g$$

Comparison of steady states

$$\text{Golden rule } f'(k^{**}) \approx n + g$$

An optimizing social planner never chooses an inefficient path. Hence, k_* should not be above k^{**}

$$\text{Ramsey } f'(k^*) \approx \rho + \frac{1}{\sigma} g$$

A comparison of the two expressions above shows that $k^* <^k **$ when

$$\rho > \left(1 - \frac{1}{\sigma}\right) g + n$$

This coincides with the condition on the next side for a maximum to exist.

Value of objective function infinite when $\rho < (1-\theta)g + n$

$$\sum_{t=0}^{\infty} \beta^t \frac{1}{\theta} [c^* A_0 (1+g)^t]^{1-\theta} L_0 (1+n)^t = \sum_{t=0}^{\infty} \text{Const.} (\beta (1+g)^{1-\theta} (1+n))^t$$

Diverges if

$$= (1+\rho)^{-1} (1+g)^{1-\theta} (1+n) > 1$$

$$-\ln(1+\rho) + (1-\theta)\ln(1+g) + \ln(1+n) > 0$$

$$\rho < (1-\theta)g + n$$

Same as condition for $k_* > k_{**}$. Optimization not meaningful when $\rho < (1-\theta)g + n$.