7 Decisions under uncertainty

Purpose of lecture:

1. Study decisions under uncertainty
2. Understand the permanent income hypothesis

7.1 Expected utility

- Define a *state of the world* as one particular realization of uncertainty (e.g., rain or sun tomorrow)
- Make two key assumptions:
  1. Suppose households have concave preferences over consumption, $u(c)$, which are separable between different states of the world. That is, utility in one state of the world does not depend on events that did not happen.
  2. Assume that households fully understand the probabilities of the risk they face
- Then households make decisions under uncertainty as if they maximize *expected utility*:
$$\max E\{u(c)\}$$
- Example: Compare two lotteries A and B. Lottery A gives c_A with probability p_A and zero otherwise. Lottery B gives c_B with probability p_B and zero otherwise. What lottery will be preferred? Household will choose A if
$$p_A u(c_A) + (1 - p_A) u(0) > p_B u(c_B) + (1 - p_B) u(0)$$
- Jensen's inequality imply that housholds prefer certainty whenever u is strictly concave (FIGURE):
$$u(E\{c\}) > E\{u(c)\}$$
- In most cases such behavior makes sense (e.g., purchase of fire insurance)

- Deviations from expected utility: Preferences for gambling and Allais paradox. These deviations imply one of two things: (1) households often misunderstand or overweight probabilities close to zero or one, or (2) preferences do not satisfy expected utility theory, for example because they have *prospect theory* preferences

7.2 A two-period consumption-savings problem

- Consider the following problem:

$$
\begin{aligned}
& \max E_1 \{u(c_1) + \beta u(c_2)\} \\
& \text{subject to} \\
c_1 &= 1 - a_2 \\
c_2 &= \tilde{w}_2 + (1+r)\, a_2,
\end{aligned}
$$

where income in period 2, $\tilde{w}_2$, is uncertain:

$$
\tilde{w}_2 = \begin{cases} 1+\varepsilon & \text{with prob. } p \\ 1-\varepsilon & \text{with prob. } 1-p \end{cases}
$$

- Substitute the budget constraints into the utility function and rewrite the problem:

$$
\begin{aligned}
& \max_{a_2} E_1 \{u(1-a_2) + \beta u(\tilde{w}_2 + (1+r)\, a_2)\} \\
=\ & \max_{a_2} \sum_{i=1}^{2} p_i \{u(1-a_2) + \beta u(\tilde{w}_2 + (1+r)\, a_2)\} \\
=\ & \max_{a_2} \{u(1-a_2) + p\beta u(1+\varepsilon + (1+r)\, a_2) \\
& + (1-p)\,\beta u(1-\varepsilon + (1+r)\, a_2)\}
\end{aligned}
$$

Differentiate w.r.t. a_2:

$$
\begin{aligned}
0 &= -u'(1-a_2) + \beta(1+r) \cdot \\
& \quad [pu'(1+\varepsilon+(1+r)\, a_2) + (1-p)\, u'(1-\varepsilon+(1+r)\, a_2)] \\
&\Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
u'(1-a_2) &= \beta(1+r) \cdot [pu'(1+\varepsilon+(1+r)\, a_2) + (1-p)\, u'(1-\varepsilon+(1+r)\, a_2)] \\
&= E_1 \{(1+r) \cdot \beta u'(\tilde{w}_2 + (1+r)\, a_2)\} \\
&\Rightarrow \\
u'(c_1) &= E_1 \{(1+r) \cdot \beta u'(c_2)\} \qquad (13)
\end{aligned}
$$

Comments:

  - This is the Euler equation under uncertainty.

- Note that solving this problem did require *rational expectations* (but not necessarily *perfect foresight*). Rational expectations requires that the household knows the future probabilities, not the realized outcomes.

- Dynamics: Suppose there are more periods, so the problem is

$$
\begin{aligned}
&\max E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \\
c_t + a_{t+1} &= \tilde{w}_t + (1+r) a_t \text{ for all } t \geq 0
\end{aligned}
$$

Write this problem in value function form:

$$
\begin{aligned}
V(a_t, w_t) &= \max \{ u(c_t) + E_t V(a_{t+1}, w_{t+1}) \} \\
&\quad \text{subject to} \\
c_t + a_{t+1} &= w_t + (1+r) a_t,
\end{aligned}
$$

where the value function V is the expected discounted utility, and a and w are the state variables. Given the function V, consumption can be expressed as a function of the same state variables: $C(a, w)$. This is the *policy rule* for consumption. It can be shown that a condition for optimality is the Euler equation, $u'(c_t) = E_t \{(1+r) \cdot \beta u'(c_{t+1})\}$, so the policy rule must satisfy the following Euler equation for any combination of the state variables (a, w):

$$
u'(C(a_t, w_t)) = E_t \{(1+r) \cdot \beta u'(C(a_{t+1}, w_{t+1}))\}.
$$

- *Detour on asset pricing*: Note that the Euler equation determines the price of a (riskless) bond q by rewriting equation (13) as follows:

$$
q \equiv \frac{1}{1+r} = E_1 \left\{ \beta \frac{u'(c_2)}{u'(c_1)} \right\}
$$

To get this, note that c_1 is known at time $t = 1$, so it is allowed to divide inside the expectation operator. This equation can also be used to find the period-1 price p_1 of any asset (whose price and dividends are stochastic and given by $\tilde{p}_2$ and $\tilde{d}_2$ next period):

$$
\begin{aligned}
p_1 \cdot u'(c_1) &= E_1 \left\{ \beta u'(c_2) \cdot \left(\tilde{p}_2 + \tilde{d}_2 \right) \right\} \\
&\Rightarrow \\
p_1 &= E_1 \left\{ \underbrace{\beta \frac{u'(c_2)}{u'(c_1)}}_{\text{stochastic discount factor}} \cdot \left(\tilde{p}_2 + \tilde{d}_2 \right) \right\}
\end{aligned}
$$

This equation is what the field of *asset pricing* in finance is all about.

7.3 Permanent income hypothesis: a special case

- We will now consider a special case of decisions under uncertainty, namely when preferences are linear-quadratic:

$$u(c) = c - \frac{a}{2}c^2$$

Note that in this case, marginal utility is given by

$$u'(c) = 1 - a \cdot c$$

- Linear-quadratic preferences imply the following Euler equation:

$$\begin{aligned} 1 - ac_t &= E_t\{(1+r)\cdot\beta(1-ac_{t+1})\} && (14) \\ &= (1+r)\beta - (1+r)\beta a E_t\{c_{t+1}\} \\ &\Rightarrow \\ c_t &= \frac{1-(1+r)\beta}{a} + (1+r)\beta \cdot E_t\{c_{t+1}\} && (15) \end{aligned}$$

- Suppose, for simplicity, that r is such that $(1+r)\beta = 1$, i.e., there is no intertemporal motive to save. In this case the Euler equation (15) becomes

$$c_t = E_t\{c_{t+1}\},$$

so the household wants to hold *expected* consumption constant.

- Recall that the budget constraint has to hold with equality. Since Ponzi schemes are ruled out $(1 + r = 1/\beta > 1)$, the discounted value of consumption must equal discounted income:

$$(1+r)a_0 + \sum_{t=0}^{\infty}\frac{w_t}{(1+r)^t} = \sum_{t=0}^{\infty}\frac{c_t}{(1+r)^t}$$

Take the expected value on both sides to obtain that consumption must equal expected income:

$$\begin{aligned} (1+r)a_0 + E_0\left\{\sum_{t=0}^{\infty}\frac{w_t}{(1+r)^t}\right\} &= E_0\left\{\sum_{t=0}^{\infty}\frac{c_t}{(1+r)^t}\right\} \\ &= \sum_{t=0}^{\infty}\frac{E_0 c_t}{(1+r)^t} \end{aligned}$$

The *law of iterated expectations* says that

$$E_0\{x_t\} = E_0\{E_1\{x_t\}\},$$

which implies that

$$E_0 c_t = E_0\{E_1 c_t\} = ... = E_0\{E_1\{E_2...\{E_{t-1}c_t\}\}\} = c_0.$$

The Euler equation can then be rewritten as

$$\begin{aligned}
(1+r)\,a_0 + E_0\left\{\sum_{t=0}^{\infty}\frac{w_t}{(1+r)^t}\right\} &= \sum_{t=0}^{\infty}\frac{E_0 c_t}{(1+r)^t} \\
&= c_0\left(\sum_{t=0}^{\infty}\frac{1}{(1+r)^t}\right) = \frac{(1+r)c_0}{r}
\end{aligned}$$

which implies that consumption is given by

$$c_0 = r\cdot\left[a_0 + \frac{1}{1+r}E_0\left\{\sum_{t=0}^{\infty}\frac{w_t}{(1+r)^t}\right\}\right],$$

Conclusion: optimal consumption is to consume a constant share r of expected lifetime wealth W. This is the *permanent income hypothesis:* $c = rW$

- By the definition of expectations, we can always write

  $$c_t = E_{t-1}\{c_t\} + \varepsilon_t,$$

  where ε_t is a stochastic variable with

  $$E_{t-1}\{\varepsilon_t\} = 0$$

  Since $c_t = E_{t-1}\{c_t\}$, we can then write

  $$c_t = c_{t-1} + \varepsilon_t,$$

  which is the *random-walk hypothesis* of Hall (1978).

- What is ε_t? Simplify by setting $r = 0$ and finite horizon, so that

  $$c_0 = \frac{1}{T+1}\cdot\left[a_0 + E_0\left\{\sum_{t=0}^{T} w_t\right\}\right]$$

  Compute c_1

  $$\begin{aligned}
  c_1 &= \frac{1}{T}\cdot\left[a_1 + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\
  &= \frac{1}{T}\cdot\left[-c_0 + a_0 + w_0 + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\
  &= \frac{1}{T}\cdot\left[-c_0 + a_0 + w_0 + E_0\left\{\sum_{t=1}^{T} w_t\right\} - E_0\left\{\sum_{t=1}^{T} w_t\right\} + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\
  &= \frac{1}{T}\cdot\left[-c_0 + (T+1)c_0 - E_0\left\{\sum_{t=1}^{T} w_t\right\} + E_1\left\{\sum_{t=1}^{T} w_t\right\}\right] \\
  &= c_0 + \frac{1}{T}\cdot\left[E_1\left\{\sum_{t=1}^{T} w_t\right\} - E_0\left\{\sum_{t=1}^{T} w_t\right\}\right] \\
  &\equiv c_0 + \varepsilon_1
  \end{aligned}$$

Here, ε_1 is the innovation in permanent income. Note:

- A permanent increase in period $t = 1$ which was unexpected in period $t = 0$: $w_t = w_0 + \Delta$ (and no further change in w_t) would give a one-for-one increase in consumption:

$$\begin{aligned} c_1 &= c_0 + \frac{1}{T} \cdot \left[E_1 \left\{ \sum_{t=1}^{T} (w_0 + \Delta) \right\} - E_0 \left\{ \sum_{t=1}^{T} w_0 \right\} \right] \\ &= c_0 + \frac{1}{T} \cdot \sum_{t=1}^{T} \Delta \\ &= c_0 + \Delta \end{aligned}$$

- A transitory (one-period) increase in period $t = 1$ which was unexpected in period $t = 0$: $w_1 = w_0 + \Delta$ (and $w_t = w_0$ thereafter) would give only a small increase in consumption:

$$\begin{aligned} c_1 &= c_0 + \frac{1}{T} \cdot \left[E_1 \left\{ w_0 + \Delta + \sum_{t=2}^{T} w_0 \right\} - E_0 \left\{ \sum_{t=1}^{T} w_0 \right\} \right] \\ &= c_0 + \frac{1}{T} \Delta \end{aligned}$$

• Note: linear-quadratic preferences exhibit certainty equivalence, in the sense that rise does not matter for savings.

• Empirical tests of the random-walk hypothesis using household-level data:

  - Souleles (1999), Johnsen et al. (2003), and others show that household-level consumption responds to (small) predictable income changes (for example tax rebates that are announced long time in advance). For example, the Bush tax rebates had big effects on consumption. This is inconsistent with PIH.
  - Using household-level data, Paxson (1993) and Hsieh (2003) show that for large predicted income changes (+/- 10%), there is no response in consumption (which is consistent with PIH).
  - How can this be true? Potential resolution: many housholds may be locked into a mortgage and may be liquidity constrained in terms of temporary expenditurs. It may be costly to change the mortgage, so one would pay such a cost (and potentially get less constrained) only if the future income change is sufficiently large. Or perhaps not all households pay attention.