Quantitative Macroeconomics: An Introduction

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Preface

In these notes I will describe how to use standard neoclassical theory to explain business cycle fluctuations.

Part I

Motivation and Data

Chapter 1

Introduction

1.1   The Questions

Business cycles are both important and, despite a large amount of economic research, still incompletely understood. While we made progress since the following quote

> The modern world regards business cycles much as the ancient Egyptians regarded the overflowing of the Nile. The phenomenon recurs at intervals, it is of great importance to everyone, and natural causes of it are not in sight. (John Bates Clark, 1898)

there is still a lot that remains to be learned. In this class we will ask, and try to at least try to partially answer the following questions

- What are them empirical characteristics of business cycles?

- What brings business cycles about?

- What propagates them?

- Who is most affected and how large would be the welfare gains of eliminating them?

- What can economic policy, both fiscal and monetary policy do in order to soften or eliminate business cycles?

- Should the government try to do so?

1.2   The Approach and the Structure of the Book

Our methodological approach will be to use economic theory and empirical data to answer these questions. We will proceed in four basic steps with our

analysis. First we will document the stylized facts that characterize business cycles in modern societies. Using real data, mostly for the US where the data situation is most favorable we will first discuss how to separate business cycle fluctuations and economic growth from the data on economic activity, especially real gross domestic product. The method for doing so is called filtering. Our stylized facts will be quantitative in nature, that is, we will not be content with saying that the growth rate of real GDP goes up and down, but we want to quantify these fluctuations, we want to document how long a business cycle lasts, whether recessions and expansions last equally long, and how large and small growth rates of real GDP or deviations from the long run growth trend are. In a second step we will then construct a theoretical business cycle model that we will use to explain business cycles. We will build up this model up in several steps, starting as a benchmark with the neoclassical growth model. At each step we will evaluate how well the model does in explaining business cycles from a quantitative point. In the process we will also have to discuss how our model is best parameterized (a process we will either call calibration or estimation, depending on the exact procedure) and how it is solved (it will turn out that we will not always be able to deal with our model analytically, but sometimes will have to resort to numerical techniques to solve the model). Into the basic growth model we will first introduce technology shocks and endogenous labor supply, which leads us to the canonical Real Business Cycle Model. Further extensions will include capital adjustment costs, two sector models and sticky prices and monopolistic competition. Once the last two elements are incorporated into the model we have arrived at the New Keynesian business cycle model. In a third step we will then evaluate the ability of the different versions of the model to generate business cycles of realistic magnitude. Once the model(s) do a satisfactory job in explaining the data, we can go on and ask normative questions. The final fourth step of our analysis will first quantify how large the welfare costs of business cycles are and then analyze, within our models, how effective monetary and fiscal policy is to tame cyclical fluctuations of the economy.

Chapter 2

Basic Business Cycle Facts

In this chapter we want to accomplish two things. First we will discuss how to distill business cycle facts from the data. The main object of macroeconomists is aggregate economic activity, that is, total production in an economy. This is usually measured by real GDP or, if one is more interested in living standards of households, by real GDP per capita or worker. But plotting the time series of real GDP we see that not only does it fluctuate over time, but it also has a secular growth trend, that it, is goes up on average. For the study of business cycle we have to purge the data from this long run trend, that is, take it out of the data. The procedure for doing so is called filtering, and we will discuss how to best filter the data in order to divide the data into a long run growth trend and business cycle fluctuations.

Second, after having de-trended the data, we want to take the business cycle component of the data and document the main stylized facts of business cycles, that is, study what are the main characteristics of business cycles. We want to document the length of a typical business cycle, whether the business cycle is symmetric, the size of deviations from the long run growth trend, and the persistence of deviations from the long-run growth trend.

2.1 Decomposition of Growth Trend and Business Cycles

In Figure 2.1 we plot the natural logarithm of real GDP for the US from 1947 to 2004. The reason we start with US data, besides it representing the biggest economy in the world, is that the data situation for this country is quite favorable. There are no obvious trend breaks, due to, say, major wars, change in the geographic structure of the country (such as the German unification), and real data are available with consistent deflation for price changes over the entire sample period. In exercises you will have the opportunity to study your country of choice, but you should be warned already that for Germany consistent data for real GDP are available only for much shorter time periods (not least because

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of the German unification).
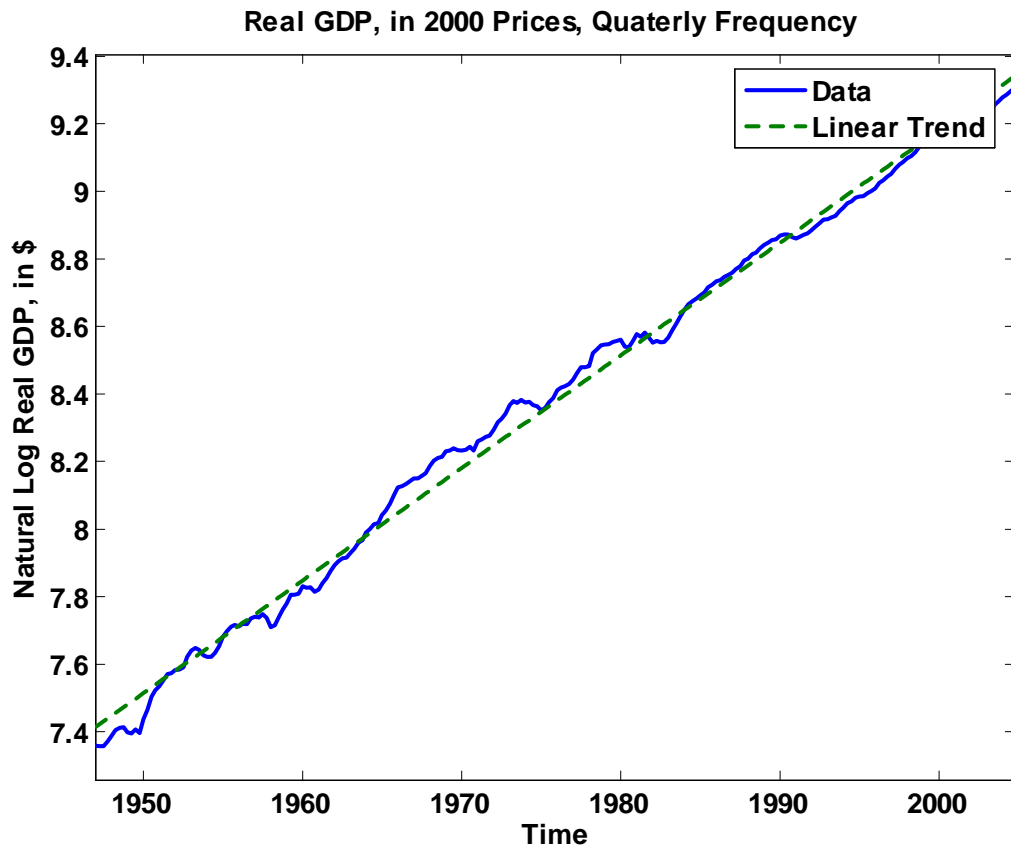
Figure 2.1: Natural Logarithm of real GDP for the US, in constant 2000 prices.

A short discussion of the data themselves. The frequency of the data is quarterly, that is, we have one observation fro real GDP in each quarter. However, the observation is for real GDP for the preceding twelve months, not just the last three months. In that way seasonal influences on GDP are controlled for. The base year for the data is 2000, that is, all numbers are in 2000 US dollars. In terms of units, the data are in billion US dollars (Milliarden). For example for 2004 real GDP is about

$$\exp(9.3) \approx \$10.000 \text{ Billion} = \$10 \text{ Trillion}$$

or about $36,000$ per capita. Finally, why would we plot the natural logarithm of the data, rather than the data themselves. Here is the reason: suppose an economic variable, say real GDP, denoted by Y, grows at a constant rate, say g,

over time. Then we have

$$Y_t = (1+g)^t Y_0 \qquad (2.1)$$

where Y_0 is real GDP at some starting date of the data, and Y_t is real GDP in period t. Now let us take logs (and whenever I say logs, I mean natural logs, that is, the logarithm with base e, where $e \approx 2.781$ is Euler's constant) of equation (2.1). This yields

$$
\begin{aligned}
\log(Y_t) &= \log\left[(1+g)^t Y_0\right] \\
&= \log(Y_0) + \log\left[(1+g)^t\right] \\
&= \log(Y_0) + t * \log\left[1+g\right]
\end{aligned}
$$

where we made use of some basic rules for logarithms.

What is important about this is that *if* a variable, say real GDP, grows at a constant rate g over time, then if we plot the logarithm of that variable it is exactly a straight line with intercept Y_0 and slope

$$\text{slope} = \log(1+g) \approx g$$

where the approximation in the last equation is quite accurate as long as g is not too large.[1] Similarly, we need not start at time $s = 0$. Suppose our data starts at an arbitrary date s (in the example $s = 1947$). Then if our data grows at a constant rate g, the figure for $\log(Y_t)$ is given by

$$\log(Y_t) = \log(Y_s) + (t - s)\log(1+g),$$

and if $s = 1947$, then

$$\log(Y_t) = \log(Y_{1947}) + (t - 1947)\log(1+g)$$

Thus, if real GDP grew at a constant rate, a plot of the natural logarithm of real GDP should be straight line, with slope equal to the constant growth rate. Figure 2.1 shows that this is not too bad of a first approximation.

In this course, however, we are mostly interested in the deviations of actual real GDP from its long run growth trend. First we want to mention that the decision what part of the data is considered a growth phenomenon and what is considered a business cycle phenomenon is somewhat arbitrary. To quote Cooley and Prescott (1995)

---

[1]The fact that $\log(1+g) \approx g$ can be seen from the Taylor series expansion of $\log(1+g)$ around $g = 0$. This yields

$$
\begin{aligned}
\log(1+g) &= \log(1) + \frac{g-0}{1} - \frac{1}{2}(g-0)^2 + \frac{1}{6}(g-0)^3 + \dots \\
&= g - \frac{1}{2}g^2 + \frac{1}{6}g^3 + \dots \\
&\approx g
\end{aligned}
$$

because the terms where g is raised to a power are small relative to g, whenever g is not too big.

Every researcher who has studied growth and/or business cycle fluctuations has faced the problem of how to represent those features of economic data that are associated with long-term growth and those that are associated with the business cycle - the deviation from the growth path. Kuznets, Mitchell and Burns and Mitchell [early papers on business cycles] all employed techniques (moving averages, piecewise trends etc.) that define the growth component of the data in order to study the fluctuations of variables around the long-run growth path defined by the growth component. Whatever choice one makes about this is somewhat arbitrary. There is no single correct way to represent these components. They ares imply different features of the same observed data.

Thus, while it is clear that "business cycle fluctuations" are the deviation of a key economic variable of interest (mostly real GDP) from a growth trend, what is unclear is how to model this growth trend. In the Figure above we made the choice of representing the long run growth trend as a function that grows at constant rate g over time. Consequently the business cycle component associated with this growth trend is given by

$$y_t = \log(Y_t) - \log(Y_t^{trend}) = \log\left(\frac{Y_t}{Y_t^{trend}}\right) = \log\left(\frac{Y_t - Y_t^{trend}}{Y_t^{trend}} + 1\right) \approx \frac{Y_t - Y_t^{trend}}{Y_t^{trend}}$$

By using logs, the deviation of the actual log real GDP from its trend roughly equals its percentage deviation from the long run growth trend. In Figure 2.2 we plot this deviation y_t from trend, when the trend is defined simply as a linear growth trend. From now on we will always use y_t to denote the business cycle component of real GDP.

The figure shows that business cycles, so defined, are characterized by fairly substantial deviations from the long run growth trend. The deviations have magnitudes of up to 10%, and they are quite persistent: if in a given quarter real GDP is above trend, it looks as if it is more likely that real GDP is above trend in the next quarter as well. We will formalize this high degree of persistence below by defining and then computing autocorrelations of real GDP. Before doing this, however, observe that if we define the trend component simply as a linear trend, the figure suggest only three basic periods. From 1947 to 1965 real GDP was below trend, from 1966 to 1982 (or 1991) it was above trend, and then it fell below trend again. According to this, the postwar US economy only had two recessions and one expansion. This seems unreasonable and is due to the fact that by defining the trend they way we have we load everything in the data this is not growing at a constant rate into the business cycle. More medium term changes in the growth rate are attributed as business cycle fluctuations. While, as we argued above the division in trend and fluctuations is always somewhat arbitrary, most business cycle researchers and practitioners take the view that the growth trend should be defined more flexibly, so that the business cycle component only captures fluctuations at the three to eight year frequency.

Real GDP, Deviation from Trend, Quaterly Frequency
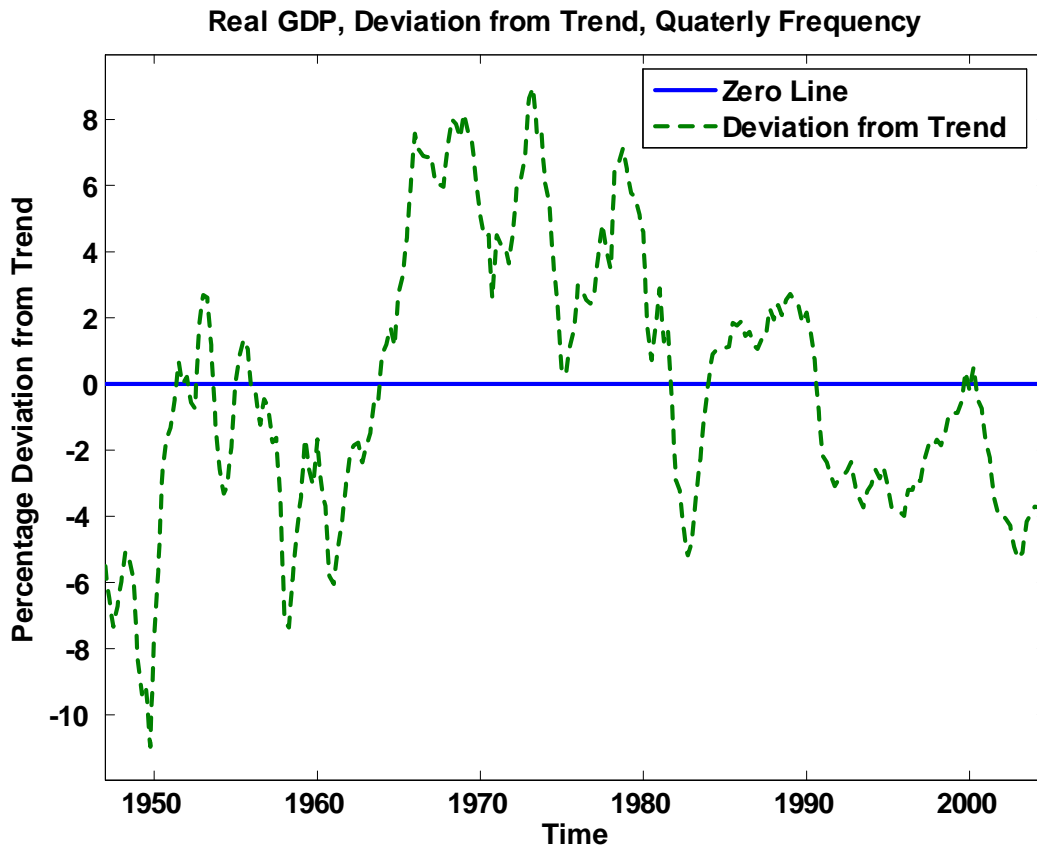
Figure 2.2: Deviations of Real GDP from Linear Trend, 1947-2004

So in practice most business cycle researchers measure business cycle fluctuations using one of three statistics: a) growth rates of real GDP, b) the cyclical component of Hodrick-Prescott filtered data, c) the data component of the appropriate frequency of a band pass filter. We will discuss the first two of these methods, and only briefly mention the third, because its understanding requires a discussion of spectral methods which you may know if you studied physics or a particular area of finance, but which I do not want to teach in this class.

Figure 2.3 plots growth rates of real GDP for the US. Remember that even though the data frequency is quarterly, these are growth rates for yearly real GDP. The average growth rate over the sample period is $3,3\%$. As a side remark, about one third of this growth is due to population and thus labor force growth, and two thirds are due to higher real GDP per capita. Note that if the log of

Growth of Real GDP, Quarterly Frequency
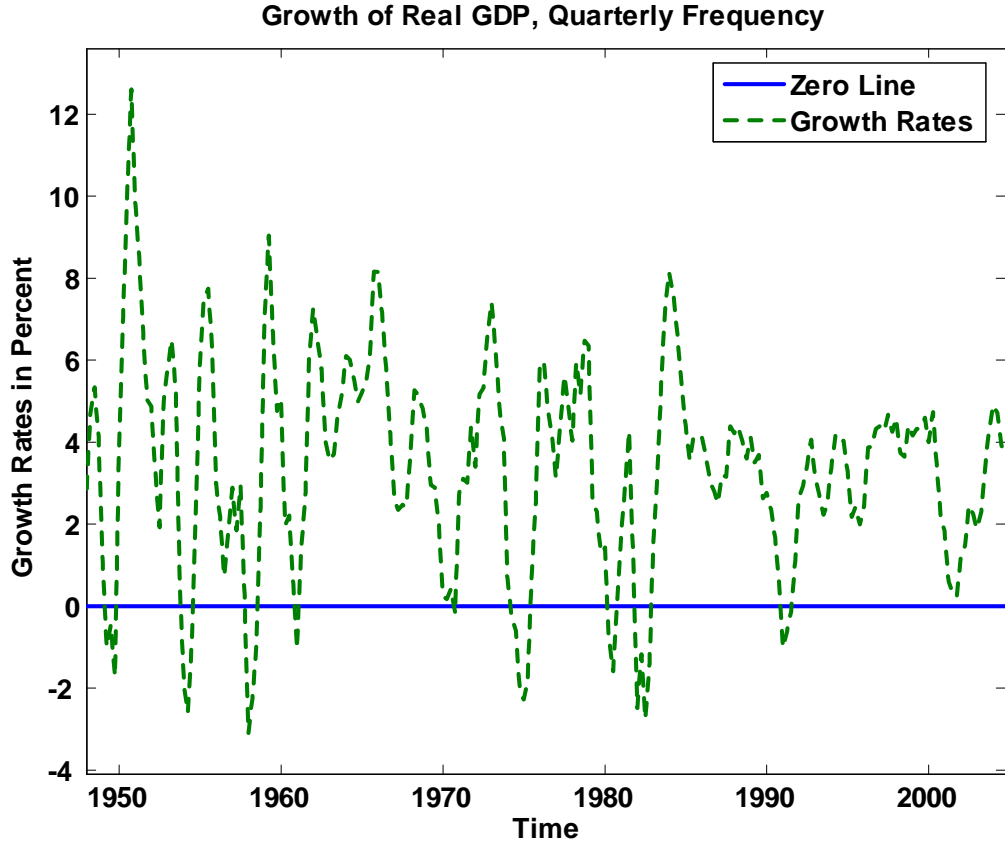
Figure 2.3: Growth Rates of Real GDP for the US, 1947-2004

real GDP would really follow exactly a linear trend, then

$$\log(Y_t) = \alpha + g(t - 1947)$$

and thus the growth rates would be given by

$$
\begin{aligned}
g_Y(t-1, t) &= \frac{Y_t - Y_{t-1}}{Y_{t-1}} \approx \log\left(\frac{Y_t - Y_{t-1}}{Y_{t-1}} + 1\right) = \log\left(\frac{Y_t}{Y_{t-1}}\right) = \log(Y_t) - \log(Y_{t-1}) \\
&= \alpha + g(t - 1947) - \alpha - g(t - 1 - 1947) = g
\end{aligned}
$$

that is, then the plot above would look like a straight line equal to the average growth rate of $3,3\%$. Note however that since the actual data do not follow this constant growth path exactly, plotting growth rates and plotting the residuals of a linear egression does *not* result in the same plot shifted by $3,3\%$ upward (compare Figure 2.3 with the trend line at $3,3\%$ and Figure 2.2). Also note

that, when dealing with $y_t = \log(Y_t)$, computing the growth rate

$$g_Y(t-1,t) = \log(Y_t) - \log(Y_{t-1}) = y_t - y_{t-1}$$

amounts to plotting the data in deviations from its value in the previous quarter. Thus effectively all variations of the data longer than one quarter are filtered out by this procedure, leaving only the very highest frequency fluctuations behind. This is why the plot looks very "jumpy", and observations in successive quarters not very correlated. We will document this fact more precisely below.

While the popular discussion mostly uses growth rates to talk about the state of the business cycle, academic economists tend to separate growth and cycle components of the data by applying a filter to the data. In fact, specifying the deterministic constant growth trend above and interpreting the deviations as cycle was nothing else than applying one such, fairly heuristic filter to the data. One filter that has enjoyed widespread popularity is the so-called Hodrick-Prescott filter, or HP-filter, for short. The goal of the filter is as before : specify a growth trend such that the deviations from that trend can be interpreted as business cycle fluctuations. Let us describe this filter in more detail and try to interpret what it does. As always we want to decompose the raw data, $\log(Y_t)$ into a growth trend $y_t^{trend} = \log(Y_t^{trend})$ and a cyclical component $y_t = \log(Y_t^{cycle})$ such that

$$\begin{aligned}\log(Y_t) &= \log(Y_t^{trend}) + \log(Y_t^{cycle}) \\ y_t &= \log(Y_t) - y_t^{trend}\end{aligned}$$

Of course the key question is how to pick y_t and y_t^{trend} from the data? The HP-filter proposes to make this decomposition by solving the following minimization problem

$$\min_{\{y_t, y_t^{trend}\}} \sum_{t=1}^{T} (y_t)^2 + \lambda \sum_{t=1}^{T} \left[(y_{t+1}^{trend} - y_t^{trend}) - (y_t^{trend} - y_{t-1}^{trend}) \right]^2 \quad (2.2)$$

subject to

$$y_t + y_t^{trend} = \log(Y_t) \quad (2.3)$$

where T is the last period of the data. Note that we are given the data $\{\log(Y_t)\}_{t=0}^{T}$, so the right hand side of 2.3 is a known and given number, for each time period. Implicit in this minimization problem is the following trade-off in choosing the trend. We may want the trend component to be a smooth function, but we also may want to make the trend component track the actual data to some degree, in order to capture also some fluctuations in the data that are of lower frequency than business cycles. These two considerations are traded off by the parameter λ. If λ is big, we want to make the terms

$$\left[(y_{t+1}^{trend} - y_t^{trend}) - (y_t^{trend} - y_{t-1}^{trend}) \right]^2$$

small. But the term

$$
\begin{aligned}
&(y_{t+1}^{trend} - y_t^{trend}) - (y_t^{trend} - y_{t-1}^{trend}) \\
=\ & \left[\log(Y_{t+1}^{trend}) - \log(Y_t^{trend})\right] - \left[\log(Y_t^{trend}) - \log(Y_{t-1}^{trend})\right] \\
=\ & g_{Y^{trend}}(t, t+1) - g_{Y^{trend}}(t-1, t)
\end{aligned}
$$

is nothing else but the change in the growth rate of the trend component. Thus a high λ makes it optimal to have a trend component with fairly constant slope. In the extreme as $\lambda \to \infty$, the weight on the second term is so big that it is optimal to set this term to 0 for all time periods, that is,

$$
\begin{aligned}
g_{Y^{trend}}(t, t+1) - g_{Y^{trend}}(t-1, t) &= 0 \\
g_{Y^{trend}}(t, t+1) &= g_{Y^{trend}}(t-1, t) = g
\end{aligned}
$$

and thus

$$
\begin{aligned}
y_t^{trend} - y_{t-1}^{trend} &= g \\
y_t^{trend} &= y_{t-1}^{trend} + g
\end{aligned}
$$

for all time periods t. But this is nothing else but our constant growth linear trend that we started with. This is, the HP-filter has the linear trend as a special case.

Now consider the other extreme, in which we value a lot the ability of the trend to follow the real data. Suppose we set $\lambda = 0$, then the objective function to minimize becomes

$$
\min_{\{y_t, y_t^{trend}\}} \sum_{t=1}^{T} (y_t)^2 \text{ subject to } y_t + y_t^{trend} = \log(Y_t)
$$

or substituting in for y_t

$$
\min_{\{y_t^{trend}\}} \sum_{t=1}^{T} \left(\log(Y_t) - y_t^{trend}\right)^2
$$

and the solution evidently is

$$
y_t^{trend} = \log(Y_t)
$$

that is, the trend is equal to the actual data series and the deviations from the trend, our business cycle fluctuations, are identically equal to zero. These extremes show that we want to pick a λ bigger than zero (otherwise there are no business cycle fluctuations and all of the data are due to the long run trend) and smaller than ∞ (so that the trend picks up some medium run variation in addition to long run growth and does not leave everything but the longest run movements to the fluctuations part).

Thus which λ to choose must be guided by our objective of filtering out business cycle fluctuations, that is, fluctuations in the data with frequency of

between three to five years. Which choice of λ accomplishes this depends crucially on the frequency of the data; for quarterly data a value of $\lambda = 1600$ is commonly used, which loads into the trend component fluctuations that occur at frequencies of roughly eight years or longer. Note that for any $\lambda \in (0, \infty)$ it is not completely straightforward to solve the minimization problem associated with the HP-filter. But luckily there exists pre-programed computer code in just about any software package to do this.

Figure 2.4 shows the trend component of the data, derived from the HP-filter with a smoothing parameter $\lambda = 1600$. That is, the figure plots y_t^{trend} against time. We observe that, in contrast to a simply constant growth trend the HP-trend captures some of the medium frequency variation of the data, which was the goal of applying the HP-filter in the first place. But the HP trend component is still much smoother than the data themselves.
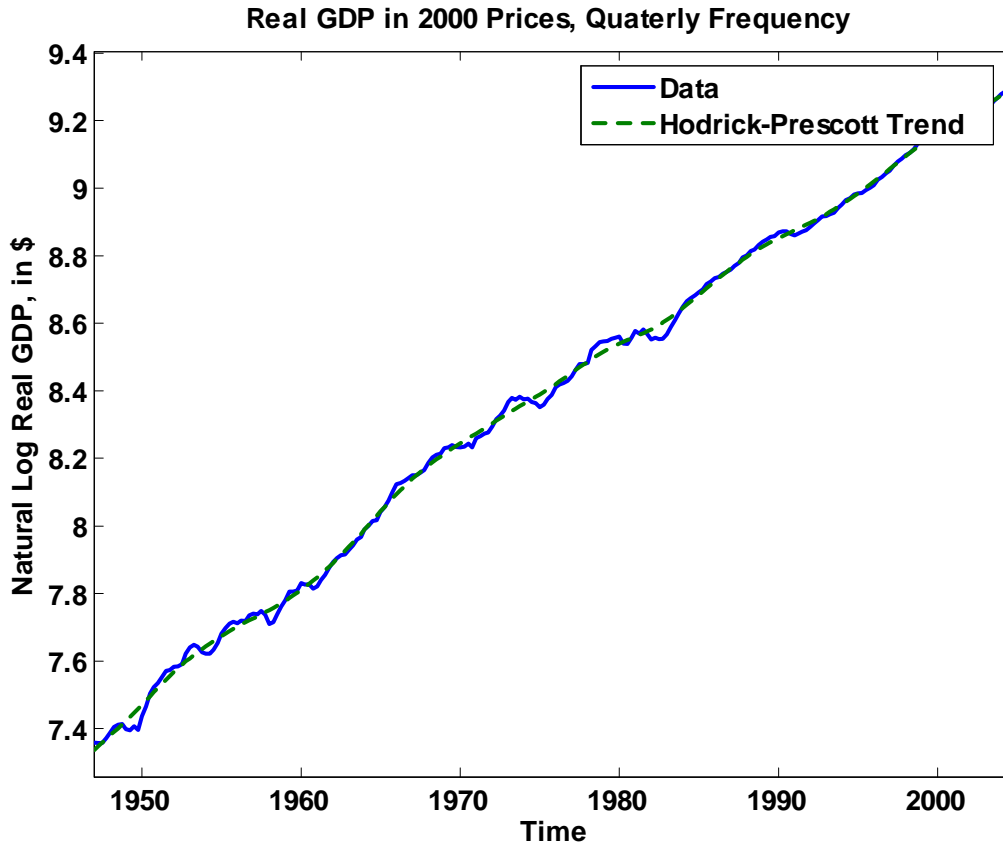
Figure 2.4: Trend Component of HP-Filtered Real GDP for the US, 1947-2004

Our true object of interest is the cyclical component that comes out of the

HP filtering Figure 2.5 displays the business cycle component of the HP-filtered data, y_t. As in Figures 2.3 and 2.2 this figure shows that the cyclical variation in real GDP can be sizeable, up to $4-6\%$ in both directions from trend. Clearly visible are the mid-seventies and early eighties recessions, both partially due to the two oil price shocks, the recession of the early 90's that cost George W. Bush's dad his job and the fairly mild (by historical standard) recent recession in 2001.
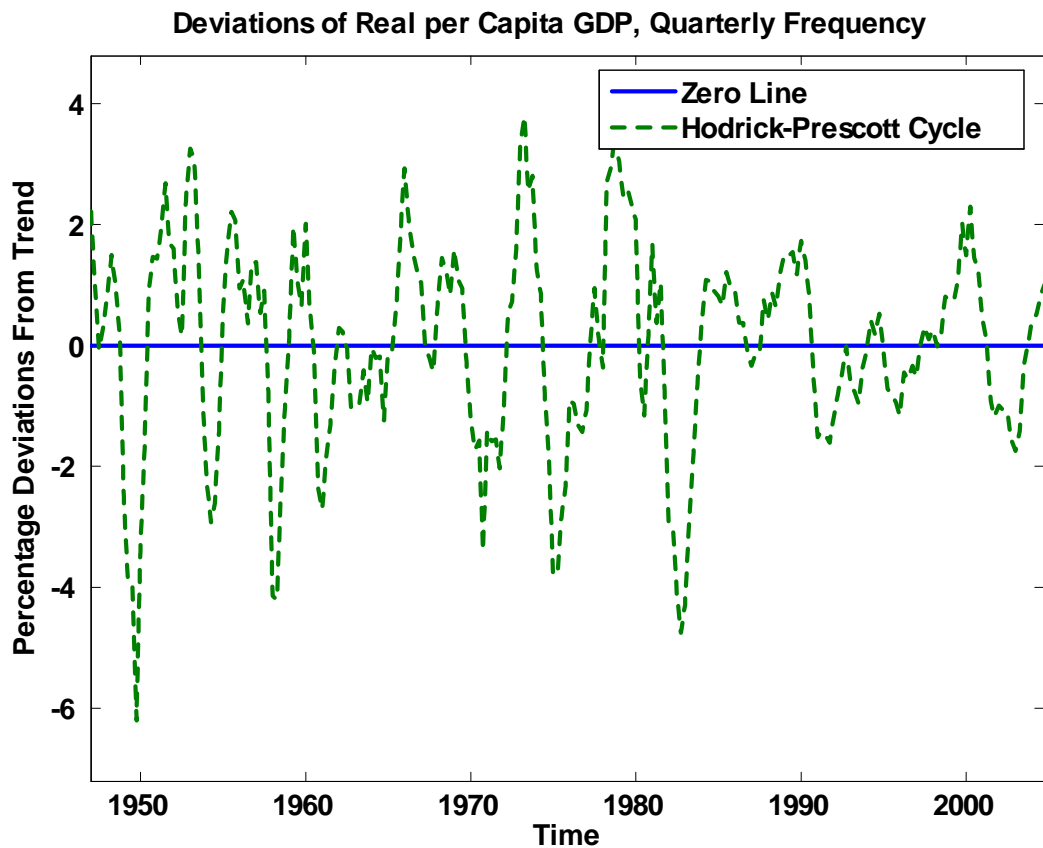
Figure 2.5: Cyclical Component of HP-Filtered Real GDP for the US, 1947-2004

But now we want to proceed with a more systematic collection of business cycle facts. We first focus on our main variable of interest, real GDP. In later chapters, once we enrich our benchmark model with labor supply and other realistic features, we will augment these facts with facts about other variables of interest.

Variable	Mean	St. Dv.	A(1)	A(2)	A(3)	A(4)	A(5)	min	max	%(>0)	%(>0,033)
Growth Rate	3.3%	2,6%	0,83	0,54	0,21	−0,09	−0,20	−3,1%	12,6%	88,6%	54,0%
HP Filter	0%	1.7%	0,84	0,60	0,32	0,08	−0,10	−6,2%	3,8%	53,9%	N/A

Table 2.1: Cyclical Behavior of Real GDP, US, 1947-2004

2.2   Basic Facts

Now that we have discussed how to be measure business cycle facts, let us
document the main regularities of business cycles. Sometimes the resulting facts
are called stylized facts, that is, facts one gets from (sophisticatedly) eyeballing
the data. These facts will be the targets of comparison for our quantitative
models to be constructed in the next part of this class. The goal of the models
is to generate business cycles of realistic magnitude, and to explain what brings
them about. In order to do so, we need empirical benchmark facts.

Table 2.1 summarizes the main stylized facts or quarterly real GDP for the
US between 1947 and 2004, both when using growth rates and when using the
cyclical component of the HP-filtered series. The mean and standard deviation
of a time series $\{x_t\}_{t=0}^T$ are defined as

$$mean(x) = \frac{1}{T} \sum_{t=0}^{T} x_t$$

$$std(x) = \left(\frac{1}{T} \sum_{t=0}^{T} (x_t - mean(x))^2 \right)^{\frac{1}{2}}$$

The autocorrelations are defined as follows

$$A(i) = corr(x_t, x_{t-1}) = \frac{\frac{1}{T-i} \sum_t (x_t - mean(x))(x_{t-i} - mean(x))}{std(x) * std(x)}$$

and measure how persistent a time series is. For time series with high first order
autocorrelation tomorrow's value is likely to be of similar magnitude as today's
value. Finally the table gives the maximum and minimum of the time series and
the fraction of observations above zero, and, for the growth rate, the fraction of
observations bigger than its mean, $3,3\%$.

We make the following observations. First, besides the fact that the mean
growth rate is $3,3\%$ whereas the cyclical component of the HP-filtered series
has a mean of 0, the main stylized facts derived from taking growth rates and
HP-filtering are about the same. They are:

1. Real GDP has a volatility of about 2% around trend, or more concretely,
   $1,7\%$ when considering the HP-filtered series.

2. The cyclical component of real GDP is highly persistent (that is, positive deviations are followed with high likelihood with positive deviations). The autocorrelation declines with the order and turns negative for the fifth order (that is if real GDP is above trend this quarter, it is more likely to be below than above trend in five quarters from now).

3. Positive deviations from trend are more likely than negative deviations from trend. This suggests that recessions are short but sharp, whereas expansions are long but gradual.

4. It is rare that the growth rate of real GDP actually becomes negative, at least for the US between 1947 and 2004.

It is an instructive exercise (that you will do with Philip's help) to carry out the same empirical exercise described in these notes for an alternative country of your choice. All you need is a sufficiently long time series for real GDP for a country (preferably seasonably adjusted), a little knowledge of MATLAB (or some equivalent software package) and a pre-programed HP filter subroutine (which for MATLAB I will give you). But now we want to start constructing our theoretical model that we will use to explain existence and magnitude of the business cycles documented above.

Part II

The Real Business Cycle (RBC) Model and Its Extensions

In this part we describe the basic real business cycle model. We start with its simplest version, in which endogenous labor supply and technology shocks are absent. That version of the model is also known in the literature as the Cass-Koopmanns neoclassical growth model. After setting up the model we will argue that the equilibrium of the model can be solved for by instead solving the maximization problem of a fictitious benevolent social planner (and we will argue why this is much easier than solving for the equilibrium directly). We then will derive and study the basic optimality conditions from this model. We will start with the explicit solution of the model by carrying out a steady state analysis, that is, we look for a special equilibrium in which the economic variables of interest (GDP, consumption, investment, the capital stock) are constant over time. Since this steady state is also the starting point of our general discussion of the solution of the model, it is a natural starting point for the analysis. We then discuss how to solve for the entire dynamic behavior of the model, which requires some mathematical and/or computational tricks. Then we quickly address how we can, in a straightforward manner and without further complications, add technological progress and population growth to the model. This will conclude the basic theoretical discussion of the benchmark model.

In order to make the model into an operational tool for business cycle analysis we have to choose parameter values that specify the elements of the model (that is, the utility function of the household and the production technology of the firms). The rigorous method for doing so within the RBC tradition is called calibration (if time permits I will transgress into a discussion of the main differences and relative advantages of calibration and formal econometric estimation). After parameterizing the model we are ready to use it. However, we will see that the benchmark model by construction does not deliver business cycles nor fluctuations of employment over time. In order to rectify this we in turn introduce a labor supply decision and stochastic (random) productivity shocks into the model.

Chapter 3

Set-Up of the Basic Model

In the benchmark model there are two types of economic actors, private households and firms. Time is discrete and the economy lasts for T periods, where $T = \infty$ (the economy lasts forever) is allowed. A typical time period is denoted by t. We now describe in turn private households and firms in this economy.

3.1 Households

Households live for T periods. All households are completely identical, and for simplicity we normalize the total number of households to 1. While this spares us to divide all macroeconomic variables by the number of people to obtain per-capita values, you should think about the economy being populated by many households that just happen to sum to 1. Where the assumption of many households is crucial is that it allows us to treat households as behaving competitively, that is, households believe that their actions do not affect market prices in the economy (because they have it in their head that there are so many households in the economy that weight in the population is negligibly small). In the simple version of the model households simply decide in each period how much of their income to consume, and how much to save for tomorrow. We assume that per period households can work a total of 1 unit of time, and since they don't care about leisure they do work all the time.

Let by c_t denote the household's consumption at time t. We assume that the household has a utility function of the form

$$
\begin{aligned}
U(c_0, c_1, \ldots, c_T) &= u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \ldots + \beta^T u(c_T) \\
&= \sum_{t=0}^{T} \beta^t u(c_t)
\end{aligned}
$$

where $\beta \in (0, 1)$ is the time discount factor. The fact that we assume $\beta < 1$ indicates that our consumer is impatient; she derives less utility from the same consumption level if that consumption occurs later, rather than earlier in life.

Sometime we will also measure the degree of the household's impatience by the time discount rate ρ. The time discount rate and the time discount factor are related via the equation

$$\beta = \frac{1}{1+\rho}$$

We will make assumptions on the period utility function u later; for now we assume that is is at least twice differentiable, strictly increasing and strictly concave (that is, $u'(c) > 0$ and $u''(c) < 0$ for all c). In applications we will often assume that the utility function is logarithmic, that is, $u(c) = \log(c)$, where log denotes the natural logarithm. This assumption is made partially because it gives us simple solutions, partly because the solution we will get has some very plausible properties.

Now that we know what people like (consumption in all periods of their life), we have to discuss what people can afford to buy. Besides working one unit of time per period and earning a wage w_t, households are born with initial assets $a_0 > 0$. Their budget constraint in period t reads as

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t$$

This equation tells us several things. First, we assume that the numeraire good is the consumption good, and normalize its price to 1. Consequently assets are real assets, that is, they pay out in terms of the consumption good, rather than in terms of money (the term money will be largely absent in this class). Similarly, w_t is the real wage and r_t is the real interest rate. The equation says that expenditures for consumption plus expenditures for purchases of assets that pay out in period $t+1$, a_{t+1}, have to equal labor income $w_t * 1$ plus the principal plus interest of assets purchased yesterday and coming due today, $(1 + r_t)a_t$. Another way of writing this is

$$c_t + a_{t+1} - a_t = w_t + r_t a_t$$

with the interpretation that labor income w_t and capital income $r_t a_t$ are spent on consumption c_t and savings $a_{t+1} - a_t$ (which is nothing else but the change in the asset position of the household between today and tomorrow). Finally note that we allow the household to purchase only one asset, a real asset with maturity of one period. In this simple model the introduction of other, more complicated assets would not change matters (i.e. the consumption allocation), but we leave the discussion of this to the many excellent asset pricing classes at Frankfurt). The household starts with assets a_0 that are given exogenously (the household is simply born with it). But what about the end of life. If we allow the household to die in debt, she would certainly decide to do so; in fact, without any limit the household optimization problem has no solution as the household would run up infinitely high debt. We assume instead that the household cannot die in debt, that is, we require $a_{T+1} \geq 0$. Since there is no point in leaving any assets unspent (the household is selfish and does not care about potential descendants, plus she knows exactly when she is dying), we immediately have $a_{T+1} = 0$. If the household lives forever, that is, $T = \infty$, then

the terminal condition for assets is slightly more complicated; one has to rule out that household debt does not grow to fast far in the future.[1] We will skip the details here; if interested, please refer to my Ph.D. lecture notes on the same issue.

This leaves us with the following household maximization problem: *given* a time path of wages and interest rates $\{w_t, r_t\}_{t=0}^T$ and initial assets a_0, the household solves

$$\max_{\{c_t, a_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$
\begin{aligned}
c_t + a_{t+1} &= w_t + (1 + r_t)a_t \\
c_t &\geq 0 \\
a_{T+1} &= 0
\end{aligned}
$$

For future reference let us derive the necessary (and if T is finite, these are also the sufficient) condition for an optimal consumption choice. First let us ignore the non-negativity constraints on consumption and the terminal condition on assets (the latter one we will use below, and it is easy to make assumptions on the utility function that guarantees $c_t > 0$ for all periods, such as $\lim_{c \to 0} u'(c) = \infty$). Setting up the Lagrangian, with λ_t denoting the Lagrange multiplier on the period t budget constraint gives

$$L = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t \left(w_t + (1 + r_t)a_t - c_t - a_{t+1} \right)$$

Taking first order conditions with respect to c_t with respect to c_{t+1} and with respect to a_{t+1} and setting them to zero yields

$$
\begin{aligned}
\beta^t u'(c_t) &= \lambda_t \\
\beta^{t+1} u'(c_{t+1}) &= \lambda_{t+1} \\
\lambda_t &= \lambda_{t+1}(1 + r_{t+1})
\end{aligned}
$$

Combining these equations yields the standard intertemporal consumption Euler equation

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1}) \tag{3.1}$$

This equation has the standard interpretation that if the household chooses consumption optimally, she exactly equates the cost from saving one more unit today (the loss of $u'(c_t)$ utils) to the benefit (saving one more unit of consumption today gives $1 + r_{t+1}$ more units of consumption tomorrow, and thus $(1 + r_{t+1}) * \beta u'(c_{t+1})$ more utils).

---

[1] A condition to rule this out is sometimes called a no Ponzi condition, in honor of a Boston business man and criminal that effectively tied to borrow without bounds. His so-called Ponzi scheme eventually exploded.

We will often be interested in a situation where the economic variables of interest, here consumption and the interest rate, are constant over time, that is, $c_t = c_{t+1} = c$ and $r_{t+1} = r$. Such a situation is often called a steady state. From the previous equation we see right away that a steady state requires

$$u'(c) = \beta(1 + r)u'(c)$$

or

$$1 = \beta(1 + r)$$

That is, in a steady state the time discount rate ρ necessarily has to equal the interest rate, $\rho = r$, because only at that interest rate will households find it optimal to set consumption constant over time (what happens if $\rho > r$ or $\rho < r$?). Of course the reverse is also true: if $r_t = \rho$ for all time periods, it follows that consumption is constant over time.

Finally we can characterize the entire dynamics of consumption, savings and asset holdings of the household, even though we may need assumptions, mathematical tricks or a computer to solve for them explicitly. Solving the budget constraint for consumption yields

$$
\begin{aligned}
c_t &= w_t + (1 + r_t)a_t - a_{t+1} \\
c_{t+1} &= w_{t+1} + (1 + r_{t+1})a_{t+1} - a_{t+2}
\end{aligned}
$$

Inserting these into equation (3.1) yields

$$u'(w_t + (1 + r_t)a_t - a_{t+1}) = \beta(1 + r_{t+1})u'(w_{t+1} + (1 + r_{t+1})a_{t+1} - a_{t+2}) \quad (3.2)$$

Remember that the household takes wages and interest rates $\{w_t, r_t\}_{t=0}^T$ as given numbers; thus the only choice variables in this equation are a_t, a_{t+1}, a_{t+2}. Mathematically, this is a second order difference equation (unfortunately a nonlinear one in general). But we have an initial condition (since a_0 is exogenously given) and a terminal condition, $a_{T+1} = 0$, so in principle we can solve this second order difference equation (in practice, as mentioned above, we either pick the utility function and thus u' well), rely on some mathematical approximations or switch on a computer and program an algorithm that solves this difference equation boundary problem. We will return to this problem below.

3.2   Firms

As with households we assume that all firms are identical and normalize the number of firms to 1. Again we still assume that firms believe to be so small that their hiring decisions do not affect the wages they have to pay their workers and the rental price they have to pay for their capital. The representative firm in each period produces the consumption good households like to eat. Let y_t denote the output of the firm of this consumption good, and n_t denote the number of workers being hired by the firm, and k_t the amount of physical capital (machines,

buildings) being used in production at period t. The production technology is described by a standard neoclassical Cobb-Douglas production function

$$y_t = A_t k_t^\alpha n_t^{1-\alpha}$$

Here A_t is a technology parameter that determines, for a given input, how much output is being produced. For now we assume that $A_t = A > 0$ is constant over time. Below we will then introduce shocks to A_t to generate business cycles. The fact that these shocks are shocks to the production technology and thus "real" shocks (as opposed to, say, monetary shocks), gives the resulting business cycle theory its name Real Business Cycle Theory. But for now the production technology is given by

$$y_t = A k_t^\alpha n_t^{1-\alpha}$$

The parameter α measures the importance of the capital input in production[2]; we will link it to the capital share of income below. In addition, when the firm uses k_t machines in period t, a fraction δ of them wear down. This process is called depreciation. Also note that this production function exhibits constant returns to scale: doubling both inputs results in doubled output.

The firm hires workers at a wage w_t per unit of time; for simplicity we also assume that the firm rents the capital it uses in the production process from the households, rather than owning the capital stock itself. This turns out to be an inconsequential assumption and makes our life easier when stating the firm's problem. Thus from now on we identify the asset the household saves with as the physical capital stock of the economy. Let the rental price per unit of capital be denoted by μ_t. Note that because of depreciation, whenever a household rents one machine to the firm, she receives $\mu_t - \delta$ as effective rental payment (since a fraction δ of the machine disappears in the production process and thus is not returned back to the household). The rental rate of capital and the real interest rate then satisfy the relation

$$r_t = \mu_t - \delta.$$

The firm takes wages and rental rates of capital as given and maximizes period by period profits (there is nothing dynamic about the firm, as it rents all inputs

---

[2]Note that α is in fact the elasticity of output with respect to the capital input. Take logs of the production function to obtain

$$\log(y_t) = \log(A) + \alpha \log(k_t) + (1-\alpha) \log(n_t)$$

and thus

$$\frac{d \log(y_t)}{d \log(k_t)} = \alpha.$$

Similarly

$$\frac{d \log(y_t)}{d \log(n_t)} = 1 - \alpha$$

period by period and sells output period by period):

$$\max_{n_t, k_t} (y_t - w_t n_t - \mu_t k_t)$$
$$\text{subject to}$$
$$y_t = A k_t^{\alpha} n_t^{1-\alpha}$$
$$k_t, n_t \geq 0$$

Again ignoring the nonnegativity constraints on inputs (given the form of the production function, these are never binding -why?) the maximization problem becomes

$$\max_{n_t, k_t} \left(A k_t^{\alpha} n_t^{1-\alpha} - w_t n_t - \mu_t k_t \right)$$

with first order conditions

$$w_t = (1-\alpha) A \left(\frac{k_t}{n_t} \right)^{\alpha}$$
$$\mu_t = \alpha A \left(\frac{k_t}{n_t} \right)^{\alpha-1}. \qquad (3.3)$$

That is, firms set their inputs such that they equate the wage rate (which they take as exogenously given) to the marginal product of labor. Likewise the rental rate of capital is equated to the marginal product of capital. An easy calculation shows that the profits of the firm equal

$$\pi_t = A k_t^{\alpha} n_t^{1-\alpha} - w_t n_t - \mu_t k_t$$
$$= A k_t^{\alpha} n_t^{1-\alpha} - (1-\alpha) A \left(\frac{k_t}{n_t} \right)^{\alpha} n_t - \alpha A \left(\frac{k_t}{n_t} \right)^{\alpha-1} k_t$$
$$= 0.$$

In fact, knowing this beforehand in the household problem above I never included profits of firms in the budget constraint. Likewise I never discussed who owns these firms. Given that their profits happen to equal zero, this is irrelevant. Note that the zero profit result does not hinge on the Cobb-Douglas production function: any constant returns to scale production function, together with price taking behavior by firms will deliver this result.

For the Cobb-Douglas production function it is also straightforward to compute the labor share and the capital share. Define the labor share as that fraction of output (GDP) that is paid as labor income, that is, the ratio of total labor income $w_t n_t$ (the product of the wage and the amount of labor used in production) to output y_t.

$$\text{labor share} = \frac{w_t n_t}{y_t}$$

A simple calculation shows that

$$\frac{w_t n_t}{y_t} = \frac{(1-\alpha) A \left(\frac{k_t}{n_t} \right)^{\alpha} * n_t}{A k_t^{\alpha} n_t^{1-\alpha}} = 1 - \alpha.$$

Similarly, capital income is given by the product of the rental rate of capital and the total amount of capital used in production, and thus the capital share equals

$$\text{capital share} = \frac{\mu_t k_t}{y_t} = \alpha.$$

Thus, if the production technology is given by a Cobb-Douglas production function, the labor share and capital share are constant over time and pinned down by the parameter α. Since we can measure capital and labor shares in the data, this relationship will be helpful in choosing a number for the parameter α when we will parameterize the economy.

3.3 Aggregate Resource Constraint

Total output produced in this economy is given by $y_t = A k_t^\alpha n_t^{1-\alpha}$. This output can be used for two purposes, for consumption and for investment (we ignore, for the moment, the government sector and assume that our economy is a closed economy). Both of these assumptions can be relaxed, but this leads to additional complications, as we will see below. Private consumption was denoted by c_t above (remember, there is only one household in this economy). Let investment be denoted by i_t. Then the resource constraint in this economy becomes

$$c_t + i_t = y_t$$

But now let's investigate investment a bit further. In the data investment takes two forms: a) the replacement of depreciated capital, replacement investment or depreciation, and b) net investment, that is, the net increase of the existing capital stock. In our model, depreciation is given by δk_t, since by assumption a fraction δ of the capital stock get broken in production. The net increase in the capital stock between today and tomorrow, on the other hand, is given by $k_{t+1} - k_t$, so that total investment equals

$$\begin{aligned} i_t &= \delta k_t + k_{t+1} - k_t \\ &= k_{t+1} - (1-\delta)k_t \end{aligned}$$

Thus the aggregate resource constraint becomes

$$c_t + k_{t+1} - (1-\delta)k_t = A k_t^\alpha n_t^{1-\alpha} \tag{3.4}$$

3.4 Competitive Equilibrium

Our ultimate goal is to study how allocations (consumption, investment output, labor etc.) in the model compare to the data. But as we have seen, these allocations are chosen by households and firms, taking prices as given. Now we have to figure out how prices (that is, wages and interest rates) are determined. We have assumed above that all agents in thee economy behave competitively,

that is, take prices as given. So it is natural to have prices be determined in what is called a competitive equilibrium. In a competitive equilibrium households and firms maximize their objective functions, subject to their constraints, and markets clear. The markets in this economy consists of a market for labor, a market for the rental of capital, and a market for goods. In a competitive equilibrium all these markets have to clear in all periods.

Let us start with the goods market. The supply of goods by firms equals its output y_t (it is never optimal for the firm to store output if there is any cost associated with it, so we'll ignore the possibilities of inventories for the time being). Demand in the goods market is given by consumption demand and investment demand of households, $c_t + i_t$. Thus the market clearing in the goods markets boils down exactly to equation (3.4). The labor market clearing condition simply states that the demand for labor by our representative firm, n_t, equals the supply of labor by our representative household. But we have assumed above that the household can and does supply one unit of labor in each period, so that the labor market clearing condition reads as

$$n_t = 1$$

Finally we have to state the equilibrium condition for the capital rental market. Firms' demand for capital rentals is given by k_t. The household's asset holdings at the beginning of period t are denoted by a_t. But physical capital is the only asset in this economy, so the assets held by households have to equal the capital that the firm desires to rent, or

$$a_t = k_t$$

All these equilibrium conditions, of course, have to hold for all periods $t = 0, \ldots, T$. But how does an equilibrium in all these markets come about? That's the role of prices (wages and interest rates): they adjust such that markets clear. We now can define a competitive equilibrium as follows:

Definition 1 *Given initial assets a_0, a competitive equilibrium are allocations for the representative household, $\{c_t, a_{t+1}\}_{t=0}^{T}$, allocations for the representative firm, $\{k_t, n_t\}_{t=0}^{T}$ and prices $\{r_t, \mu_t, w_t\}_{t=0}^{T}$ such that*

1. *Given $\{r_t, w_t\}_{t=0}^{T}$, the household allocation solves the household problem*

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{subject to}$$

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t$$
$$c_t \geq 0$$
$$a_{T+1} = 0$$

2. *Given $\{\mu_t, w_t\}_{t=0}^T$, with $\mu_t = r_t - \delta$ for all $t = 0, \ldots, T$ the firm allocation solves the firm problem*

$$\max_{n_t, k_t} (y_t - w_t n_t - \mu_t k_t)$$

$$\text{subject to}$$

$$y_t = A k_t^\alpha n_t^{1-\alpha}$$

$$k_t, n_t \geq 0$$

3. *Markets clear: for all $t = 0, \ldots, T$*

$$c_t + k_{t+1} - (1 - \delta)k_t = A k_t^\alpha n_t^{1-\alpha}$$

$$n_t = 1$$

$$a_t = k_t$$

Note that the definition of equilibrium is completely silent about how the equilibrium prices come about; the definition simply states that at the equilibrium prices markets clear. Also note something surprising: there are $3(T + 1)$ market clearing conditions (there are $T + 1$ time periods and 3 markets open per period), but we have only $2(T + 1)$ prices that can be used to bring about market clearing (wages w_t and interest rates r_t for each period $t = 0, \ldots, T$; note that μ_t does not count, since it always equals $\mu_t + r_t - \delta$). But it turns out that whenever, in a given period, two markets clear, then the third market clears automatically. In fact, this is an important general result in General Equilibrium Theory (as the research field that deals with competitive equilibria and their properties is called). The result is commonly referred to as Walras' law.

Theorem 2 *Suppose that at prices $\{r_t, \mu_t, w_t\}_{t=0}^T$, allocations $\{c_t, a_{t+1}\}_{t=0}^T$ and $\{k_t, n_t\}_{t=0}^T$ solve the household problem and the firm problem and suppose that all periods t the labor and the asset markets clear, $n_t = 1$ and $a_t = k_t$ for all t. Then the goods market clears, for all t.*

Proof. Since the allocation solves the household problem, it has to satisfy the household budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t$$

By market clearing in the asset market, $a_t = k_t$ and $a_{t+1} = k_{t+1}$, and thus

$$c_t + k_{t+1} = w_t + (1 + r_t)k_t$$

or

$$c_t + k_{t+1} - k_t = w_t + r_t k_t$$

But since $r_t = \mu_t - \delta$, we have

$$c_t + k_{t+1} - k_t = w_t + (\mu_t - \delta)k_t$$

or

$$
\begin{aligned}
c_t + k_{t+1} - (1 - \delta)k_t & = w_t + \mu_t k_t \\
& = 1 * w_t + \mu_t k_t \\
& = n_t w_t + \mu_t k_t \qquad (3.5)
\end{aligned}
$$

where the last equality uses the labor market clearing condition $n_t = 1$. But from the first order conditions of the firms' problem (remember we assumed that the allocation solves the firms problem)

$$
\begin{aligned}
w_t & = (1 - \alpha)A \left(\frac{k_t}{n_t} \right)^\alpha \\
\mu_t & = \alpha A \left(\frac{k_t}{n_t} \right)^{\alpha - 1}
\end{aligned}
$$

and thus

$$
\begin{aligned}
w_t n_t + \mu_t k_t & = (1 - \alpha)A \left(\frac{k_t}{n_t} \right)^\alpha n_t + \alpha A \left(\frac{k_t}{n_t} \right)^{\alpha - 1} k_t \\
& = (1 - \alpha)A k_t^\alpha n_t^{1-\alpha} + \alpha A k_t^\alpha n_t^{1-\alpha} \\
& = A k_t^\alpha n_t^{1-\alpha}. \qquad (3.6)
\end{aligned}
$$

Combining (3.5) and (3.6) yields

$$
c_t + k_{t+1} - (1 - \delta)k_t = A k_t^\alpha n_t^{1-\alpha}
$$

that is, the goods market clearing condition. ∎

3.5 Characterization of Equilibrium

What we want to do with this model is to characterize equilibrium allocations and assess how well the model describes reality. We already found the optimality condition of the household as

$$
u'(w_t + (1 + r_t)a_t - a_{t+1}) = \beta(1 + r_{t+1})u'(w_{t+1} + (1 + r_{t+1})a_{t+1} - a_{t+2})
$$

Now we can make use of further optimality conditions to simplify matters further. First, we realize that from the asset market clearing condition we have $k_t = a_t, k_{t+1} = a_{t+1}$ and $k_{t+2} = a_{t+2}$. Then we know that

$$
\begin{aligned}
w_t & = (1 - \alpha)A \left(\frac{k_t}{n_t} \right)^\alpha = (1 - \alpha)A k_t^\alpha \\
r_t & = \mu_t - \delta = \alpha A (k_t)^{\alpha - 1} - \delta
\end{aligned}
$$

and similar results hold for w_{t+1} and r_{t+1}. Inserting all this in the Euler equation of the household yields

$$
\begin{aligned}
& u'((1 - \alpha)A k_t^\alpha + (1 + \alpha A (k_t)^{\alpha - 1} - \delta)k_t - k_{t+1}) \\
= \; & \beta(1 + \alpha A (k_{t+1})^{\alpha - 1} - \delta)u'((1 - \alpha)A k_{t+1}^\alpha + (1 + \alpha A (k_{t+1})^{\alpha - 1} - \delta)k_{t+1} - k_{t+2})
\end{aligned}
$$

or

$$u'(Ak_t^\alpha + (1-\delta)k_t - k_{t+1}) = \beta(1+\alpha A (k_{t+1})^{\alpha-1} - \delta)u'(Ak_{t+1}^\alpha + (1-\delta)k_{t+1} - k_{t+2})$$
(3.7)

Arguably this is a mess, but this equation has as arguments only (k_t, k_{t+1}, k_{t+2}). All the remaining elements in this equation are the parameters $(\alpha, A, \delta, \beta)$ and of course the derivative of the utility function that needs to be specified (and by doing so we may need additional parameters). But mathematically speaking, this now really is simply a second order difference equation (there are no prices there anymore). What is more, we have an initial condition $k_0 = a_0$, equal to some pre-specified number, and $k_{T+1} = a_{T+1} = 0$. Below we will study techniques (mostly numerical in nature) to solve an equation like this.

Chapter 4

Social Planner Problem and Competitive Equilibrium

In this section we will show something that, at first sight, should be fairly surprising. Namely that we can solve for the allocation of a competitive equilibrium by solving the maximization problem of a benevolent social planner. In fact, this is a very general principle that often applies (and as such is called a theorem, in fact, two theorems, namely the first and second fundamental welfare theorems of general equilibrium). Envision a social planner that can tell agents in the economy (households, firms) what to do, i.e. how much to consume, how much to work, how much to produce etc. The social planner is benevolent, that is, likes the households in the economy and thus maximizes their lifetime utility function. The only constraints the social planner faces are the physical resource constraints of the economy (even the social planner cannot make consumption out of nothing).

4.1   The Social Planner Problem

The problem of the social planner is given by

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} - (1 - \delta)k_t = A k_t^\alpha n_t^{1-\alpha}$$
$$c_t \geq 0, 0 \leq n_t \leq 1$$
$$k_0 > 0 \text{ given}$$

We make two simple observations before characterizing the optimal solution to the social planner problem. First, households only value consumption in their utility function, and don't mind working. Since more work means higher output,

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and thus higher consumption today, or via higher investment tomorrow, it is always optimal to set $n_t = 1$. Second, we have not imposed any constraint on k_{t+1}, but evidently $k_{t+1} < 0$ can never happen since then production is not well-defined (try to raise a negative number to some power $\alpha \in (0, 1)$ and see what your pocket calculator tells you). In addition, even $k_{t+1} = 0$ can be ruled out under fairly mild condition on the utility function, since $k_{t+1} = 0$ means that output in period $t+1$ is zero, thus consumption in that period is zero and $k_{t+2} = 0$ and so forth. Thus it is never optimal to set $k_{t+1} = 0$ for any time period, unless the household does not mind too much consuming 0 (in all our applications this will never happen). Exploiting these facts the social planners problem becomes

$$\max_{\{c_t, k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} - (1 - \delta)k_t = Ak_t^\alpha$$
$$c_t \geq 0 \text{ and } k_0 > 0 \text{ given}$$

Note that this maximization problem is an order of magnitude less complex than solving for a competitive equilibrium, because in the later we have to solve maximization problems of households and firms and find equilibrium prices that lead to market clearing. The social planner problem is a simple maximization problem, although that problem still has many choice variables, namely $2(T+1)$, which is a big number as T becomes large. Thus it would be really useful to know that by solving the social planner problem we have in fact also found the competitive equilibrium.

4.2   Characterization of Solution

The Lagrangian, ignoring the non-negativity constraints for consumption (which will not be binding under the appropriate conditions on the utility function), is given by

$$L = \sum_{t=0}^T \beta^t u(c_t) + \sum_{t=0}^T \lambda_t \left[Ak_t^\alpha - c_t - k_{t+1} + (1 - \delta)k_t \right]$$

The first order conditions, set to 0, are

$$\frac{\partial L}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = 0$$
$$\frac{\partial L}{\partial c_{t+1}} = \beta^{t+1} u'(c_{t+1}) - \lambda_{t+1} = 0$$
$$\frac{\partial L}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1} \left[\alpha Ak_{t+1}^{\alpha-1} + (1 - \delta) \right] = 0$$

Rewriting these conditions yields

$$
\begin{aligned}
\beta^t u'(c_t) &= \lambda_t \\
\beta^{t+1} u'(c_{t+1}) &= \lambda_{t+1} \\
\lambda_{t+1} \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right] &= \lambda_t
\end{aligned}
$$

and thus

$$
\begin{aligned}
\beta^t u'(c_t) &= \lambda_t = \lambda_{t+1} \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right] = \beta^{t+1} u'(c_{t+1}) \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right] \\
u'(c_t) &= \beta u'(c_{t+1}) \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right] \tag{4.1}
\end{aligned}
$$

which is, of course, the intertemporal Euler equation. The social planner equates the marginal rate of substitution of the representative household between consumption today and tomorrow, $\frac{u'(c_t)}{\beta u'(c_{t+1})}$ to the the marginal rate of transformation between today and tomorrow. Letting the household consume one unit of consumption less today allows for one more unit investment and thus one more unit of capital tomorrow. But this additional unit of capital yields additional production equal to the marginal product of capital, $\alpha A k_{t+1}^\alpha$, and after production $1 - \delta$ units of the capital are still left over. Thus the marginal rate of transformation between consumption today and tomorrow equals $\alpha A k_{t+1}^\alpha + (1-\delta)$.

Making use of the resource constraints

$$
\begin{aligned}
c_t &= A k_t^\alpha - k_{t+1} + (1-\delta)k_t \\
c_{t+1} &= A k_{t+1}^\alpha - k_{t+2} + (1-\delta)k_{t+1}
\end{aligned}
$$

the Euler equation becomes

$$
u'(A k_t^\alpha - k_{t+1} + (1-\delta)k_t) = \beta u'(A k_{t+1}^\alpha - k_{t+2} + (1-\delta)k_{t+1}) \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right] \tag{4.2}
$$

Comparing equations (4.2) and (3.7) we see that they are exactly the same. This observation will be the basis of our proof of the two welfare theorems. Note again that this is a second order difference equation, and we have an initial condition, because k_0 is given to us. As long as $T < \infty$ we also have a terminal condition, since it is evidently optimal for the planner to set $k_{T+1} = 0$ (why invest into capital that is productive in the period after the world ends). If $T = \infty$ things are a bit more complicated as there is no last period.

4.3   The Welfare Theorems

Now we can state the two fundamental theorems of welfare economics,. We will restrict the proof to the case that $T < \infty$ since this avoids some tricky issues with infinite-dimensional optimization (if $T = \infty$, the social planner as well as the household in the competitive equilibrium has to choose infinitely many consumption levels and capital stocks).

Theorem 3 *(First Welfare Theorem) Suppose we have a competitive equilibrium with allocation $\{c_t, k_{t+1}\}_{t=0}^T$. Then the allocation is socially optimal (in the sense that it solves the social planner problem).*

Proof. Since $\{c_t, k_{t+1}\}_{t=0}^T$ is part of a competitive equilibrium, it has to satisfy the necessary conditions for household and firm optimality. We showed that this implies that the allocation solves the Euler equation (3.7). But then the allocation satisfies the necessary and sufficient conditions (why are the conditions sufficient?) of the social planners problem (4.2). ∎

Theorem 4 *(Second Welfare Theorem) Suppose an allocation $\{c_t, k_{t+1}\}_{t=0}^T$ solves the social planners problem and hence is socially optimal. Then there exist prices $\{r_t, \mu_t, w_t\}_{t=0}^T$ that, together with the allocation $\{c_t, k_{t+1}\}_{t=0}^T$ and $\{n_t, a_{t+1}\}_{t=0}^T$, where $n_t + 1$ and $a_{t+1} = k_{t+1}$ for all t, forms a competitive equilibrium.*

Proof. The proof is by construction. First we note that all market clearing conditions for a competitive equilibrium are satisfied (the labor market and asset market equilibrium conditions by construction, the goods market clearing condition since the allocation, by assumption, solves the social planner problem and thus satisfies the resource feasibility condition, for all t. Now construct prices as functions of the allocation, as follows

$$
\begin{aligned}
w_t &= (1-\alpha)A \left(\frac{k_t}{n_t} \right)^{\alpha} \\
r_t &= \alpha A \left(\frac{k_t}{n_t} \right)^{\alpha-1} - \delta \\
\mu_t &= r_t + \delta
\end{aligned}
$$

It remains to be shown that at these prices the allocation solves the firms' and households' maximization problem. Looking at the firms first order conditions, they are obviously satisfied. And the necessary and sufficient conditions for the households' maximization problem were shown to boil down to (3.7), which the allocation satisfies, since it solves the social planner problem and hence the conditions (4.2). ∎

These two results come in handy, because they allow us to solve the much simpler social planner problem and be sure to have automatically solved for the competitive equilibrium, the ultimate object of interest, also.

4.4 Appendix: More Rigorous Math

More specifically, here is how the logic works in precise mathematical terms.

1. Suppose the social planner problem has a unique solution (since it is a finite-dimensional maximization with a strictly concave objective function and a convex constraint set). The second theorem tells us that we can make this solution a competitive equilibrium, and the proof of the second theorem even tells us how to construct the prices.

2. Can there be another competitive equilibrium allocation? No, since the first theorem tells us that that other equilibrium allocation would also be a solution to the social planner problem, contradicting the fact that this problem has a unique solution.

The last steps also should indicate to you that the case $T = \infty$ is harder to deal with mainly because arguing for uniqueness of the solution to the social planners problem and for the sufficiency of the Euler equations for optimality is harder if the problem is infinite-dimensional.

Chapter 5

Steady State Analysis

A steady state is a competitive equilibrium or a solution to the social planners problem where all the variables are constant over time. Let (c^*, k^*) denote the steady state consumption level and capital stock. Then, if the economy starts with the steady state capital stock $k_0 = k^*$, it never leaves that steady state. And even if it starts at some $k_0 \neq k^*$, it may over time approach the steady state (and once it hits it, of course it never leaves again. We will see below that the steady state may also an important starting point of the dynamic analysis of the model; if there is hope that the solution of the model never gets too far away from the steady state, one can linearize the dynamic system around the steady state and hope to obtain a good approximation to the optimal decisions. More on this below.

5.1 Characterization of the Steady State

Since the previous section showed that we can interchangeably analyze socially optimal and equilibrium allocations, let us focus on equation (4.2), or equivalently, equation (4.1)

$$u'(c_t) = \beta u'(c_{t+1}) \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right] \tag{5.1}$$

In the steady state we require $c_t = c_{t+1} = c^*$ and $k_{t+1} = k^*$. But then $u'(c_t) = u'(c_{t+1})$ and thus

$$1 = \beta \left[aA \left(k^*\right)^{\alpha-1} + (1-\delta) \right]$$

or, remembering $\beta + \frac{1}{1+\rho}$,

$$\rho = aA \left(k^*\right)^{\alpha-1} - \delta$$
$$\rho + \delta = aA \left(k^*\right)^{\alpha-1} \tag{5.2}$$

This rule for choosing the steady state capital stock is sometimes called the modified golden rule: the optimal steady state capital stock is such that the

associated marginal product of capital equals the depreciation rate plus the time discount rate. We will see very soon why this is called the *modified* golden rule. Obviously we can solve for the steady state capital stock explicitly as

$$k^* = \left(\frac{\rho + \delta}{aA}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{\alpha A}{\rho + \delta}\right)^{\frac{1}{1 - \alpha}}$$

That is, the optimal steady state capital stock is the higher the more productive the production technology (that is, the higher A), the more important capital is relative to labor in the production process (the higher is α), the lower is the depreciation rate of capital (the lower δ) and the lower the impatience of individuals (the lower ρ). Also note that the steady state capital stock is completely independent of the utility function of the household (as long as it is strictly concave, why?)

The steady state consumption level can now be determined from the resource constraint (3.4)

$$c_t + k_{t+1} - (1 - \delta)k_t = Ak_t^\alpha$$

In steady state this becomes

$$
\begin{aligned}
c^* + \delta k^* &= A\left(k^*\right)^\alpha \\
c^* &= A\left(k^*\right)^\alpha - \delta k^*
\end{aligned}
$$

that is, steady state consumption equals steady state output minus depreciation, since in the steady state capital is constant and thus net investment $k_{t+1} - k_t$ is equal to 0.

5.2   Golden Rule and Modified Golden Rule

Now we can also see why the modified golden rule is called that way. In the steady state, consumption equals

$$c = A\left(k\right)^\alpha - \delta k.$$

Let k^g denote the (traditional) golden rule capital stock that maximizes steady state consumption, i.e. the capital stock that solves

$$\max_k A\left(k\right)^\alpha - \delta k$$

Taking first order conditions and setting to 0 yields

$$\alpha A \left(k^g\right)^{\alpha - 1} = \delta \text{ or} \tag{5.3}$$

$$k^g = \left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1 - \alpha}} \tag{5.4}$$

Let c^g denote the (traditional) golden rule consumption level

$$c^g = A\left(k^g\right)^\alpha - \delta k^g.$$

We have the following

Proposition 5 *The modified golden rule capital stock and consumption level are lower than the (traditional) golden rule capital stock and consumption level, strictly so whenever $\rho > 0$ (and or course $A > 0, \alpha > 0, \delta > 0$.*

Proof. Obviously

$$k^g = \left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1-\alpha}} \geq k^* = \left(\frac{\alpha A}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}$$

with strict inequality if $\rho > 0$. But k^g is the unique capital stock that maximizes steady state consumption, whereas k^* does not maximize steady state consumption. Thus, evidently, $c^g \geq c^*$, with strict inequality if $k^g > k^*$ (i.e. if $\rho > 0$). ∎

What is the intuition for the result? Why is the perfectly benevolent planner choosing a steady state capital and consumption level lower than the one that would maximize steady state consumption. Note that the planners objective is to maximize lifetime utility, not lifetime consumption. And since the household is impatient if $\rho > 0$, the planner should take this into account by letting the household consume more today, as the expense of lower steady state consumption in the future.

Chapter 6

Dynamic Analysis

Now we want to analyze how the economy, from an arbitrary starting condition k_0, evolves over time. Again we exploit the welfare theorems and go for the solution of the social planner problem directly. It is somewhat easier to do this analysis for the case $T = \infty$, that is, for the case in which the economy runs forever. The reason is simple: if the economy ends at a finite T, we know that $k_{T+1} = 0$. On the other hand it is never optimal to have $k_t = 0$ for any time $t \leq T$, because otherwise consumption from that point on would have to equal 0, which is never optimal under some fairly mild conditions on the utility function.[1] Therefore it is unlikely that in the finite time case the economy will settle down to a steady state with constant capital and consumption. In the case $T = \infty$ this will happen, as we will see below.

We already know that if by coincidence we have $k_0 = k^*$, then the capital stock would remain constant over time in the $T = \infty$ case, so would be the interest rate, consumption and all other variables of interest. This can be seen from the Euler equation of the social planners problem, equation (4.2):

$$
\begin{aligned}
u'(Ak_t^\alpha - k_{t+1} + (1-\delta)k_t) &= \beta u'(Ak_{t+1}^\alpha - k_{t+2} + (1-\delta)k_{t+1}) \left[\alpha Ak_{t+1}^{\alpha-1} + (1-\delta)\right] \\
&= \frac{u'(Ak_{t+1}^\alpha - k_{t+2} + (1-\delta)k_{t+1}) \left[\alpha Ak_{t+1}^{\alpha-1} + (1-\delta)\right]}{1+\rho} \quad (6.1)
\end{aligned}
$$

Plugging in $k_t = k_{t+1} = k_{t+2} = k^*$ we see that this equation holds (simply realize that $\frac{\alpha A(k^*)^{\alpha-1} + (1-\delta)}{1+\rho} = 1$ and that the left had side for $k_t = k_{t+1} = k^*$ equals the right hand side for $k_{t+1} = k_{t+2} = k^*$), so that setting $k_t = k^*$ for all periods is a solution to the social planners problem (in fact the only solution, since the optimal solution to the social planner problem is unique, because it is

---

[1] Besides strict concavity (i.e. $u''(c) < 0$) what is needed is a so-called Inada condition

$$
\lim_{c \to 0} u'(c) = \infty,
$$

that is, one assumes that at zero consumption the marginal utility from consuming the first unit is really large. The utility function $u(c) = \log(c)$ satisfies this assumption.

a maximization problem with strictly concave objective function and a convex constraint set).

But what happens if we start with $k_0 \neq k^*$? We assume that $k_0 > 0$, since otherwise there is no production, no consumption and no investment in period 0 or in any period from that point on. What we are looking for is a sequence of numbers $\{k_t\}_{t=1}^{\infty}$ that solves equation (6.1). This is in general hard to do, so let's try to simplify. The economy starts with initial capital k_0 and the social planner has to decide how much of this to let the agent consume, c_0 and how much to accumulate for tomorrow, k_1. But in period 1 the planner has exactly the same problem: given the current capital stock k_1, how much to let the agent consume, c_1 and how much to accumulate for tomorrow, k_2. This is in fact a general principle (which one can make very general using the general theory of dynamic programming): we can find our solution $\{k_t\}_{t=1}^{\infty}$ by finding the unknown function g giving $k_{t+1} = g(k_t)$. If we would know this function, then we can determine how the capital stock evolves over time by starting with the given k_0, and then

$$
\begin{aligned}
k_1 &= g(k_0) \\
k_2 &= g(k_1) \\
k_3 &= g(k_2)
\end{aligned}
$$

and so forth. Of course the challenge is to find this function g. There are two approaches to do this. For some examples we can guess a particular form of g and then verify that our guess was correct. Second, and this is a very general approach that (almost) always works and for which the use of a computer comes in handy, one can try to turn equation (6.1) into a linear equation, which one then can always solve.

6.1   An Example with Analytical Solution

Suppose that the utility function is logarithmic, $u(c) = \log(c)$ (and thus $u'(c) = \frac{1}{c}$) and also assume that capital completely depreciates after one period, that is $\delta = 1$. Then equation (6.1) becomes

$$
\frac{1}{Ak_t^{\alpha} - k_{t+1}} = \frac{\beta \alpha A k_{t+1}^{\alpha-1}}{Ak_{t+1}^{\alpha} - k_{t+2}} \tag{6.2}
$$

Remember that we look for a function g telling us how big k_{t+1} is, given today's capital stock k_t. Now lets make a wild guess (maybe not so wild: let us guess that the social planner finds it optimal to take output Ak_t^{α} and split it between consumption c_t and capital tomorrow, k_{t+1} in fixed proportions, independent of the level of the current capital stock and thus output of the economy (remember the Solow model?). That is, let us guess that

$$
k_{t+1} = g(k_t) = sAk_t^{\alpha}
$$

where s is a fixed number. Thus, applying the same guess for period $t+1$ yields

$$k_{t+2} = g(k_{t+1}) = sAk_{t+1}^{\alpha}.$$

Given these guesses we have

$$
\begin{aligned}
Ak_t^{\alpha} - k_{t+1} &= (1-s)Ak_t^{\alpha} \\
Ak_{t+1}^{\alpha} - k_{t+2} &= (1-s)Ak_{t+1}^{\alpha}
\end{aligned}
$$

Using these two equations in (6.2) yields

$$
\begin{aligned}
\frac{1}{(1-s)Ak_t^{\alpha}} &= \frac{\beta\alpha Ak_{t+1}^{\alpha-1}}{(1-s)Ak_{t+1}^{\alpha}} \\
&= \frac{\beta\alpha}{(1-s)k_{t+1}}
\end{aligned}
$$

and thus

$$k_{t+1} = \beta\alpha Ak_t^{\alpha}$$

But we had guessed

$$k_{t+1} = sAk_t^{\alpha}$$

and thus with $s = \beta\alpha$ and therefore $k_{t+1} = \beta\alpha Ak_t^{\alpha}$ equation (6.2) is satisfied for *every* k_t. That is, no matter how big k_t is, if we set

$$
\begin{aligned}
k_{t+1} &= \beta\alpha Ak_t^{\alpha} \\
k_{t+2} &= \beta\alpha Ak_{t+1}^{\alpha}
\end{aligned}
$$

the Euler equation is satisfied. Thus we found exactly what we were looking for, namely a function that tells us that if the planner gets into the period with capital k_t, how much does she take out of the period. And since we know the initial capital stock k_0, we can compute the entire sequence of capital stocks from period 0 on (of course always conditional on parameter values β, α). A computer is very good carrying out such a calculation, as you will see in one of Philip's tutorials. Finally, it is obvious that from the resource constraint

$$c_t = Ak_t^{\alpha} - k_{t+1} + (1-\delta)k_t$$

we can compute consumption over time, once we know how capital evolves over time.

We now want to briefly analyze the dynamics of the capital stock, starting from k_0 and being described by the so-called policy function

$$k_{t+1} = \beta\alpha Ak_t^{\alpha}$$

There are two steady states of this policy function in which $k_{t+1} = k_t = k$. The first is, trivially, $k = 0$. The second is our steady state from above, $k = k^*$, solving

$$
\begin{aligned}
k^* &= \beta\alpha A\left(k^*\right)^{\alpha} \text{ or} \\
k^* &= (\beta\alpha A)^{\frac{1}{1-\alpha}} = \left(\frac{\alpha A}{\rho+1}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$
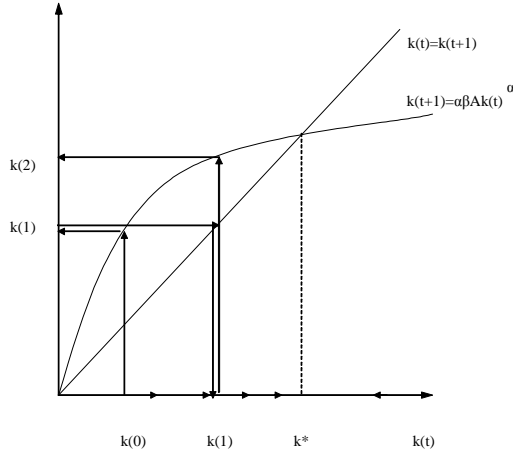
Figure 6.1: Dynamics of the Neoclassical Growth Model

(remember that $\delta = 1$ was assumed above).

In order to describe the dynamics of the capital stock it is best to plot the policy function. Figure 6.1 does exactly that. In addition it contains the line $k_{t+1} = k_t$. If we start from the initial capital stock k_0, the graph gives $k_1 = g(k_0)$ on the y-axis. Going back to the $k_{t+1} = k_t$ line gives k_1 on the x-axis. Then the graph gives $k_2 = g(k_1)$. Going again to the $k_{t+1} = k_t$ line gives k_2 on the x-axis. One can continue this procedure. The important feature from this figure is that if $k_0 < k^*$, over time k_t increases and approaches the steady state k^*. This is true for any $k_0 > 0$ with $k_0 < k^*$. In contrast, if $k_0 > k^*$, then the capital stock approaches the steady state from above. In either case there is monotonic convergence to the steady state: the capital stock is either monotonically increasing or decreasing and over time approaches the steady state, which justifies our focus in the previous section. Technically speaking, the unique positive steady state is globally asymptotically stable.

6.2 Linearization of the Euler Equation

In general the Euler equation is not a linear equation, and thus we can't easily solve it. But what we can do is to make the Euler equation linear by approximating the nonlinear equation linearly around its steady state. Roughly speaking this is nothing else but doing a Taylor series approximation around the steady state of the model, which we have solved before. The resulting approximated Euler equation is linear, and hence easy to solve for the policy function g of interest. Keep in mind, however, that we only solve for an approximate solution (unless of course we assume quadratic utility and linear production, as above). While there are numerical methods to solve for the exact nonlinear solution, these are not easy to implement and thus not discussed in this class.

Let c^*, k^*, y^* denote the steady state values of consumption, capital and output. We have already discussed above how to find this steady state. The idea is to replace the nonlinear Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) \left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta) \right] \tag{6.3}$$

where

$$
\begin{aligned}
c_t &= A k_t^\alpha - k_{t+1} + (1-\delta)k_t \\
c_{t+1} &= A k_{t+1}^\alpha - k_{t+2} + (1-\delta)k_{t+1}
\end{aligned}
$$

with a linear version, because linear equations are much easier to solve.

6.2.1 Preliminaries

There is one crucial tool for turning the nonlinear Euler equation into a linear equation: Taylor's theorem. This theorem states that we can approximate a function $f(x)$ around a point a by writing

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \dots$$

Note that the approximation is only exact by using an infinite number of terms. But in order to make a nonlinear equation into a linear equation we will use the approximation

$$f(x) \approx f(a) + f'(a)(x-a) \tag{6.4}$$

and hope we will not make too big of an error. Note however that for all $x \neq a$ we will make an error; how severe this error is depends on the application and can only exactly be answered for examples in which we know the true solution. In our application we will use as point of approximation a the steady state of the economy. As long as the economy does not move too far away from the steady state, we can hope that the approximation in (6.4) is accurate.

Similarly, for a function of two variables we have

$$f(x,y) \approx f(a,b) + \frac{\partial f(a,b)}{\partial x}(x-a) + \frac{\partial f(a,b)}{\partial y}(y-b) \tag{6.5}$$

Finally, often researchers prefer to express deviations of a variable x_t from its steady state x^* not in its deviation $(x_t - x^*)$, but in its percentage deviation $\hat{x}_t = \frac{x_t - x^*}{x_t}$, since this is easier to interpret. For future reference note that

$$\log(x_t) - \log(x^*) = \log\left(\frac{x_t}{x^*}\right) = \log\left(\frac{x_t - x^*}{x^*} + 1\right) \approx \hat{x}_t = \frac{x_t - x^*}{x^*}$$

Also for future reference note that

$$x_t - x^* = \frac{x_t - x^*}{x^*} x^* = \hat{x}_t x^*$$

We will use these facts repeatedly below.

6.2.2   Doing the Linearization

We want to linearize

$$
\begin{align}
u'(c_t) &= \beta u'(c_{t+1})\left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\right] \quad \text{where} \tag{6.6}\\
c_t &= A k_t^\alpha - k_{t+1} + (1-\delta)k_t \tag{6.7}\\
c_{t+1} &= A k_{t+1}^\alpha - k_{t+2} + (1-\delta)k_{t+1} \tag{6.8}
\end{align}
$$

Let's do this slowly for the first time, since this procedure is prone to making mistakes (believe me, I know what I am talking about). We start with equation (6.7), which can be rewritten as

$$0 = A k_t^\alpha - k_{t+1} + (1-\delta)k_t - c_t \tag{6.9}$$

Let us do term by term. First, let's linearize the term $A k_t^\alpha$ around the steady state k^*. This yields, using (6.4),

$$
\begin{align}
A k_t^\alpha &\approx A(k^*)^\alpha + \alpha A(k^*)^{\alpha-1} * (k_t - k^*)\\
&= A(k^*)^\alpha + \alpha A(k^*)^{\alpha-1} * k^* \hat{k}_t\\
&= A(k^*)^\alpha + \alpha A(k^*)^\alpha \hat{k}_t
\end{align}
$$

Note that the only variable here is $\hat{k}_t$, since k^* is the steady state capital stock, and hence a fixed number, and the rest is just a bunch of parameters. Now consider the term $-k_{t+1}$; again using (6.4) we get

$$-k_{t+1} \approx -k^* - (k_{t+1} - k^*) = -k^* - k^* \hat{k}_{t+1}$$

where in this case the linear approximation is exact, since the function $f(k_{t+1}) = -k_{t+1}$ is linear to start with. Similarly

$$
\begin{align}
(1-\delta)k_t &\approx (1-\delta)k^* + (1-\delta)k^* \hat{k}_t\\
-c_t &\approx -c^* - c^* \hat{c}_t
\end{align}
$$

and thus the linear approximation of (6.9) reads as

$$
\begin{aligned}
0 &\approx A\left(k^*\right)^\alpha + \alpha A\left(k^*\right)^\alpha \hat{k}_t - k^* - k^*\hat{k}_{t+1} + (1-\delta)k^* + (1-\delta)k^*\hat{k}_t - c^* - c^*\hat{c}_t \\
&= A\left(k^*\right)^\alpha - k^* + (1-\delta)k^* - c^* + \alpha A\left(k^*\right)^\alpha \hat{k}_t - k^*\hat{k}_{t+1} + (1-\delta)k^*\hat{k}_t - c^*\hat{c}_t.
\end{aligned}
$$
$$(6.10)$$

Finally realize that, at the steady state (c^*, k^*), the resource constraint (6.9) holds. Thus we can simplify (6.10) to (replacing the $\approx$ with an equality sign)

$$
\begin{aligned}
0 &\approx \alpha A\left(k^*\right)^\alpha \hat{k}_t - k^*\hat{k}_{t+1} + (1-\delta)k^*\hat{k}_t - c^*\hat{c}_t \\
c^*\hat{c}_t &= \alpha A\left(k^*\right)^\alpha \hat{k}_t - k^*\hat{k}_{t+1} + (1-\delta)k^*\hat{k}_t \\
&= \left[\alpha A\left(k^*\right)^{\alpha-1} + (1-\delta)\right]k^*\hat{k}_t - k^*\hat{k}_{t+1}
\end{aligned}
$$
$$(6.11)$$

which is a linear equation with the three variables $(\hat{c}_t, \hat{k}_t, \hat{k}_{t+1})$. Using exactly the same logic we can linearize (6.8) to obtain

$$
\begin{aligned}
c^*\hat{c}_{t+1} &= \alpha A\left(k^*\right)^\alpha \hat{k}_{t+1} - k^*\hat{k}_{t+2} + (1-\delta)k^*\hat{k}_{t+1} \\
&= \left[\alpha A\left(k^*\right)^{\alpha-1} + (1-\delta)\right]k^*\hat{k}_{t+1} - k^*\hat{k}_{t+2}
\end{aligned}
$$
$$(6.12)$$

The really painful equation is equation (6.6)

$$
\begin{aligned}
0 &= -u'(c_t) + \beta u'(c_{t+1})\left[\alpha A k_{t+1}^{\alpha-1} + (1-\delta)\right] \\
&= -u'(c_t) + (1-\delta)\beta u'(c_{t+1}) + \beta u'(c_{t+1})\alpha A k_{t+1}^{\alpha-1}
\end{aligned}
$$
$$(6.13)$$

Again proceeding term by term we have, using (6.4),

$$
\begin{aligned}
-u'(c_t) &\approx -u'(c^*) - u''(c^*)(c_t - c^*) \\
&= -u'(c^*) - u''(c^*)c^*\hat{c}_t
\end{aligned}
$$

and

$$(1-\delta)\beta u'(c_{t+1}) = (1-\delta)\beta u'(c^*) + (1-\delta)\beta u''(c^*)c^*\hat{c}_{t+1}$$

The last term is more complicated, since it involves a function of two variables, (c_{t+1}, k_{t+1}). Using (6.5) we have

$$
\begin{aligned}
&\beta u'(c_{t+1})\alpha A k_{t+1}^{\alpha-1} \\
&\approx \beta u'(c^*)\alpha A\left(k^*\right)^{\alpha-1} + \beta u''(c^*)\alpha A\left(k^*\right)^{\alpha-1}(c_{t+1} - c^*) + \beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-2}(k_{t+1} - k^*) \\
&= \beta u'(c^*)\alpha A\left(k^*\right)^{\alpha-1} + \beta u''(c^*)\alpha A\left(k^*\right)^{\alpha-1}c^*\hat{c}_{t+1} + \beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-1}\hat{k}_{t+1}
\end{aligned}
$$

Inserting all terms into (6.13) and re-grouping yields

$$
\begin{aligned}
0 &= -u'(c^*) + (1-\delta)\beta u'(c^*) + \beta u'(c^*)\alpha A\left(k^*\right)^{\alpha-1} \\
&\quad -u''(c^*)c^*\hat{c}_t + (1-\delta)\beta u''(c^*)c^*\hat{c}_{t+1} + \beta u''(c^*)\alpha A\left(k^*\right)^{\alpha-1}c^*\hat{c}_{t+1} + \beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-1}\hat{k}_{t+1}
\end{aligned}
$$

But the entire first line of this expression equals zero, since equation (6.13) holds in the steady state ($c_t = c_{t+1} = c^*, k_{t+1} = k^*$). Thus the linearized version of equation (6.13) becomes

$$
\begin{aligned}
u''(c^*)c^*\hat{c}_t &= (1-\delta)\beta u''(c^*)c^*\hat{c}_{t+1} + \beta u''(c^*)\alpha A\left(k^*\right)^{\alpha-1}c^*\hat{c}_{t+1} + \beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-1}\hat{k}_{t+1} \\
&= \left[(1-\delta) + \alpha A\left(k^*\right)^{\alpha-1}\right]\beta u''(c^*)c^*\hat{c}_{t+1} + \beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-1}\hat{k}_{t+1} \qquad (6.14)
\end{aligned}
$$

which is again a linear function in $(\hat{c}_t, \hat{c}_{t+1}, \hat{k}_{t+1})$. Arguably the system of equations (6.6), (6.7) and (6.8) is a mess, but it is a linear mess. Inserting (6.6) and (6.7) into (6.8) yields

$$
\begin{aligned}
&u''(c^*)\left\{\left[\alpha A\left(k^*\right)^{\alpha-1} + (1-\delta)\right]k^*\hat{k}_t - k^*\hat{k}_{t+1}\right\} \\
&= \left[(1-\delta) + \alpha A\left(k^*\right)^{\alpha-1}\right]\beta u''(c^*) * \\
&\quad \left\{\left[\alpha A\left(k^*\right)^{\alpha-1} + (1-\delta)\right]k^*\hat{k}_{t+1} - k^*\hat{k}_{t+2}\right\} \\
&\quad + \beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-1}\hat{k}_{t+1}
\end{aligned}
$$

Collecting terms yields

$$
\begin{aligned}
&\left\{u''(c^*)\left[\alpha A\left(k^*\right)^{\alpha-1} + (1-\delta)\right]\right\}k^*\hat{k}_t \\
&-\left\{\begin{array}{c}u''(c^*)k^* + \left[(1-\delta) + \alpha A\left(k^*\right)^{\alpha-1}\right]\beta u''(c^*)\left[\alpha A\left(k^*\right)^{\alpha-1} + (1-\delta)\right]k^* \\ +\beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-1}\end{array}\right\}\hat{k}_{t+1} \\
&+\left[(1-\delta) + \alpha A\left(k^*\right)^{\alpha-1}\right]\beta u''(c^*)k^*\hat{k}_{t+2} \\
&= 0
\end{aligned}
$$

or

$$
c_1\hat{k}_t + c_2\hat{k}_{t+1} + c_3\hat{k}_{t+2} = 0 \qquad (6.15)
$$

where the three constants (c_1, c_2, c_3) are given by

$$
\begin{aligned}
c_1 &= u''(c^*)k^*/\beta < 0 \\
c_2 &= -\left\{u''(c^*)k^* + u''(c^*)k^*/\beta + \beta u'(c^*)\alpha(\alpha-1)A\left(k^*\right)^{\alpha-1}\right\} > 0 \\
c_3 &= u''(c^*)k^* < 0
\end{aligned}
$$

since in the steady state $\left[(1-\delta) + \alpha A\left(k^*\right)^{\alpha-1}\right]\beta = 1$. The nice thing about equation (6.15) is that this, in contrast to (6.3), is a *linear* second order difference equation, which we know very well how to solve.

Again, what we are looking for is a policy function giving $\hat{k}_{t+1} = \hat{g}(\hat{k}_t)$. Let us again guess that this function takes the form

$$
\begin{aligned}
\hat{k}_{t+1} &= \hat{g}(\hat{k}_t) = \hat{s}\hat{k}_t \text{ and thus} \\
\hat{k}_{t+2} &= \hat{s}\hat{k}_{t+1} = \hat{s}^2\hat{k}_t
\end{aligned}
$$

where $\hat{s}$ is a constant to be determined. Inserting the guessed policy function into equation (6.15) yields[2]

$$
\begin{aligned}
c_1 \hat{k}_t + c_2 \hat{s} \hat{k}_t + c_3 \hat{s}^2 \hat{k}_t &= 0 \text{ or} \\
c_1 + c_2 \hat{s} + c_3 \hat{s}^2 &= 0 \\
\hat{s}^2 + \frac{c_2}{c_3} \hat{s} + \frac{c_1}{c_3} &= 0 \\
\hat{s}^2 - \gamma_1 \hat{s} + \gamma_2 &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_2 &= \frac{c_1}{c_3} = \frac{1}{\beta} \\
\gamma_1 &= -\frac{c_2}{c_3} = 1 + \frac{1}{\beta} + \frac{\beta u'(c^*)\alpha(\alpha-1)A(k^*)^{\alpha-1}}{u''(c^*)k^*}
\end{aligned}
$$

This is simply a quadratic equation in $\hat{s}$ with the two solutions

$$
\hat{s}_{1,2} = \frac{\gamma_1}{2} \pm \left(\frac{\gamma_1^2}{4} - \gamma_2 \right)^{\frac{1}{2}}
$$

There are some good news here. First we note that, since

$$
\gamma_1 > \gamma_2 + 1 \tag{6.16}
$$

we have that

$$
\frac{\gamma_1^2}{4} - \gamma_2 > \frac{1}{4}(1+\gamma_2)^2 - \gamma_2 = \frac{1}{4}\left[(1+\gamma_2)^2 - 4\gamma_2\right] = \frac{1}{4}(1-\gamma_2)^2 \geq 0 \tag{6.17}
$$

and thus both roots are real. Second we note that

$$
\hat{s}_1 * \hat{s}_2 = \frac{\gamma_1^2}{4} - \frac{\gamma_1^2}{4} + \gamma_2 = \frac{1}{\beta}
$$

and since $\gamma_1 > 0$ we know that both roots are positive. The bigger root satisfies, using (6.16) and (6.17)

$$
\hat{s}_1 = \frac{\gamma_1}{2} + \left(\frac{\gamma_1^2}{4} - \gamma_2 \right)^{\frac{1}{2}} > \frac{\gamma_1}{2} + \frac{1}{2}(\gamma_2 - 1) = \frac{\gamma_1 + \gamma_2 - 1}{2} > \frac{2\gamma_2}{2} = \gamma_2 = \frac{1}{\beta}
$$

and thus the smaller root satisfies

$$
\hat{s}_2 = \frac{\gamma_1}{2} - \left(\frac{\gamma_1^2}{4} - \gamma_2 \right)^{\frac{1}{2}} < 1.
$$

---

[2] The procedure of parameterizing the policy function by a coefficient $\hat{s}$ and then solving for it is sometimes called the method of undetermined coefficients.

What have we accomplished? We guessed that the optimal policy function satisfied

$$\hat{k}_{t+1} = \hat{s}\hat{k}_t$$

and found that such a function indeed satisfies the log-linearized version of the Euler equation, as long as either $\hat{s} = \hat{s}_1$ or $\hat{s} = \hat{s}_2$. But what is the right choice of $\hat{s}$? Remember that

$$\hat{k}_t = \frac{k_t - k^*}{k^*} \text{ and thus}$$
$$k_t = (1 + \hat{k}_t)k^*.$$

Also remember that we are dealing with the case $T = \infty$. Therefore we only have the initial condition k_0 given, but no terminal condition. Now suppose that we select $\hat{s}_1 > 1$ and suppose $k_0 < k^*$, so that $\hat{k}_0 < 0$, that is we start with our capital stock below the steady state. But our policy function tells us

$$\hat{k}_1 = \hat{s}_1 k_0$$
$$\hat{k}_2 = \hat{s}_1 k_1 = \hat{s}_1^2 k_0 \text{ and in general}$$
$$\hat{k}_t = \hat{s}_1^t k_0 \tag{6.18}$$

Since $\hat{s}_1 > 1$, as t gets larger and larger, $\hat{s}_1^t$ grows exponentially towards infinity. Consequently $\hat{k}_t$ becomes more and more negative. At some point in time we have $\hat{k}_t < -1$, and thus

$$k_t = (1 + \hat{k}_t)k^* < 0 \tag{6.19}$$

which can never be optimal. A similar argument shows that if the economy starts with $k_0 > k^*$ and thus $\hat{k}_0 > 0$, then $\hat{k}_t \to \infty$ as time goes to infinity. Thus k_t increases over time $(k_{t+1} > k_t)$ and $k_t \to \infty$ with time. But again this cannot be optimal since

$$c_t = Ak_t^\alpha + (1 - \delta)k_t - k_{t+1} < Ak_t^\alpha - \delta k_t < 0$$

if $k_t > \left(\frac{A}{\delta}\right)^{\frac{1}{1-\alpha}}$, which happens for large enough t (as long as $\alpha < 1$). It can simply not be optimal to drive the capital stock to infinity, since at some point simply replacing the depreciated capital requires all of current production, and nothing is left for consumption. Thus with $\hat{s} = \hat{s}_1$ the capital stock either becomes negative or explodes over time which cannot be the optimal policy of the social planner. Therefore we can discard this root, and focus on the other root $\hat{s}_2 \in (0, 1)$. For this root we have that $\hat{s}_1^t \to 0$ as t becomes large. Thus from (6.18) we see that $\hat{k}_t$ goes to 0 over time, and from (6.19) we see that k_t converges to the steady state k^* over time. Since the optimal policy of the linearized problem is

$$\hat{k}_{t+1} = \hat{s}\hat{k}_t$$

with $\hat{s} \in \{\hat{s}_1, \hat{s}_2\}$ and we have discarded the case $\hat{s} = \hat{s}_1$, our optimal policy is

$$\hat{k}_{t+1} = \hat{s}_2 \hat{k}_t$$

where $\hat{s}_2 \in (0,1)$ is a fixed number that of course depends on the parameter values of the economy. The dynamics of the capital stock of the economy (and thus output, consumption and the like) is described as monotonic convergence to the steady state, from above if $k_0 > k^*$ and from below if $k_0 < k^*$. It looks very much like the dynamics in figure 6.1, with the difference that the policy function in the figure was nonlinear, whereas now it is linear (in percentage deviations). The root $\hat{s}_1$ is sometimes called the unstable root, because it leads the capital stock to explode or implode. The root $\hat{s}_2$ is called the stable root, since it leads to monotonic convergence to the steady state. A situation where both roots are real and exactly one is stable is a very desirable situation, because then we have one, and only one, optimal policy function. The neoclassical growth model has this desired property.

Two final remarks. First, note that the policy function is linear in percentage deviations from the steady state. Of course it is easy to derive the policy function in capital levels. We have

$$\hat{k}_{t+1} = \hat{s}_2 \hat{k}_t \tag{6.20}$$
$$\frac{k_{t+1} - k^*}{k^*} = \hat{s}_2 * \frac{k_t - k^*}{k^*}$$
$$k_{t+1} - k^* = \hat{s}_2 \left(k_t - k^* \right)$$
$$k_{t+1} = (1 - \hat{s}_2)k^* + \hat{s}_2 k_t \tag{6.21}$$

Second, I again want to stress that the so-derived policy function is only an approximation to the true policy function. We can see this from the example in the last section where we derived the true policy function as

$$k_{t+1} = \beta \alpha A k_t^\alpha \tag{6.22}$$

Whatever the value for $\hat{s}_2$ is (of course we can calculate it, it is a messy function of the parameters of the model), the approximation in (6.21) is not equal to the true policy function (6.22), unless of course we are in the steady state: $k_t = k^*$. Note however, since $\hat{k}_t \approx \log(k_t) - \log(k^*)$, we have from (6.20)

$$\log(k_{t+1}) \approx (1 - \hat{s}_2) \log(k^*) + \hat{s}_2 \log(k_t).$$

On the other hand, taking logs of (6.22) yields

$$\log(k_{t+1}) = \log(\beta \alpha A) + \alpha \log(k_t).$$

Thus, for the particular example our linear (in percentage deviations from the steady state) approximation is very accurate; in fact the only approximation (which we know is very good for small deviations from steady state) is due to the fact that

$$\frac{k_t - k^*}{k^*} \approx \log(k_t) - \log(k^*)$$

Thus for the parameterization in the previous section ($\delta = 1$, $u(c) = \log(c)$), we would expect $\hat{s}_2 = \alpha$. It is a useful exercise to verify this.

Obviously all other variables of interest (consumption c, output y and investment i) can be easily computed once we know capital. For this one can either use the exact equations, e.g.

$$c_t = A k_t^{\alpha} + (1 - \delta) k_t - k_{t+1}$$

or the linear approximation, if one prefers this

$$c^* \hat{c}_t = \left[\alpha A \left(k^* \right)^{\alpha - 1} + (1 - \delta) \right] k^* \hat{k}_t - k^* \hat{k}_{t+1}.$$

6.3   Analysis of the Results

6.3.1   Plotting the Policy Function

One way to represent the results from the previous section is to simply plot $\hat{k}_{t+1} = g(\hat{k}_t)$ against $\hat{k}_t$ as in figure 6.1. Usually the dynamics of the capital stock can be deduced from such a plot already fairly completely, at least in a model without stochastic shocks as we have discussed so far.

6.3.2   Impulse Response Functions

The idea of an impulse response function is to plot what happens to our variables of interest in response to an exogenous shock to the economy, conditional on the economy being in the steady state before the shock. While in a model without stochastic shocks this is a bid strange, one can still do it. Suppose that the economy at time $t = 0$ is in the steady state ($k_0 = k^*, \hat{k}_0 = 0$), and then at the beginning of period 1 for some reason the capital stock gets reduced by 1% (a terroristic attack, say). We can use the policy function $\hat{k}_{t+1} = g(\hat{k}_t)$ to deduce what happens to the capital stock over time:

$$
\begin{aligned}
\hat{k}_0 &= 0, k_0 = k^* \\
\hat{k}_1 &= -0.01, k_1 = 0.99 k^* \\
\hat{k}_2 &= -0.01 \hat{s}_2, k_2 = (1 - 0.01 \hat{s}_2) k^* \text{ and in general} \\
\hat{k}_t &= -0.01 \hat{s}_2^{t-1}, k_t = (1 - 0.01 \hat{s}_2^{t-1}) k^*
\end{aligned}
$$

A plot of the sequence $\{k_t\}_{t=0}^{T}$ or $\{\hat{k}_t\}_{t=0}^{T}$ is called an impulse response function, because it plots the response of the capital stock to the initial impulse of a shock over time. Obviously we can do exactly the same with the policy function of the nonlinear problem, if we can possibly solve it.

Figure 6.2 plots an impulse response function for capital. The impulse is a 1% reduction of the capital stock, and the policy function used is $\hat{k}_{t+1} = 0.333 \hat{k}_t$. We observe the quick convergence of the model back to its steady state.

6.3.3   Simulations

In a deterministic model (i.e. a model without stochastic shocks) a simulation is something very similar to an impulse response. In a stochastic model the
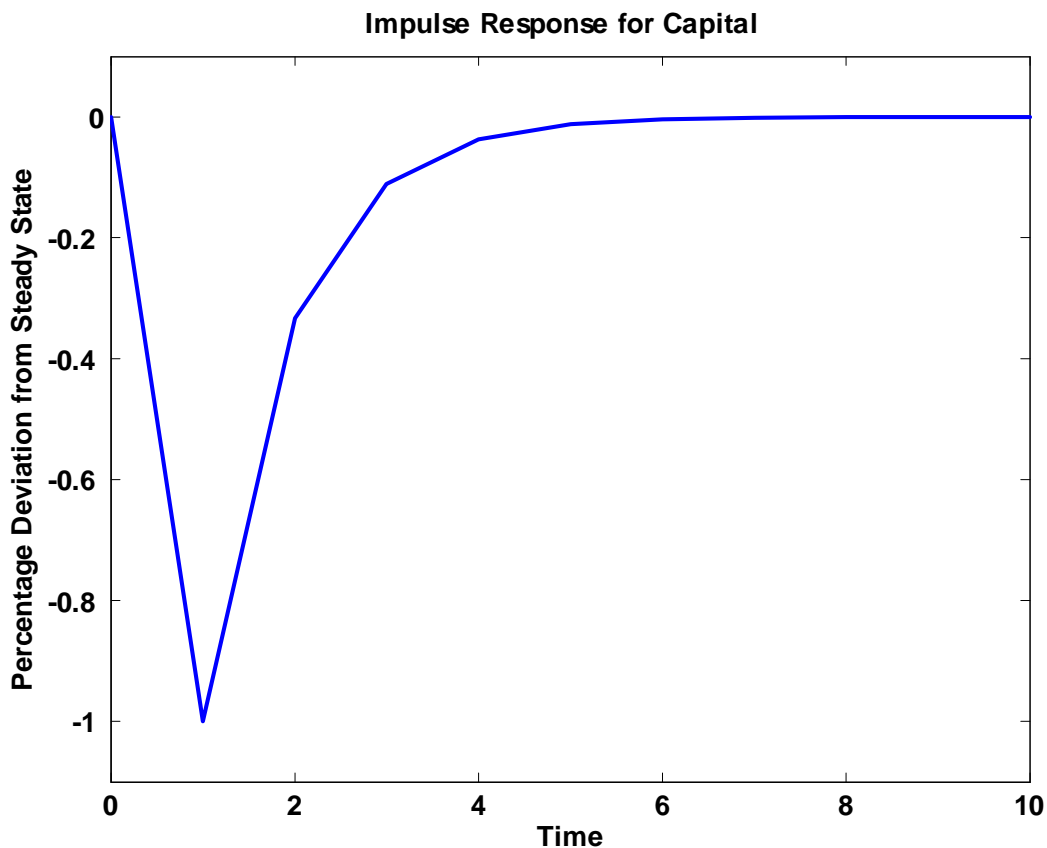
Figure 6.2: Impulse Response Function for Capital

difference between the two ways to analyze the results of the model will be more different. In a simulation one picks some initial condition of the economy, k_0, and then uses the policy function g to simulate a long sequence of capital stocks according to $k_{t+1} = g(k_t)$, that is

$$
\begin{aligned}
k_1 &= g(k_0) \\
k_2 &= g(k_1)
\end{aligned}
$$

and so forth. Of course one can again plot the sequence $\{k_t\}$ against time. Alternatively, one can compute statistics of interest of these artificial data (for example the standard deviation, autocorrelation and so forth). In order to assess how good the model performs empirically one can then compare the statistics derived from the artificial model data to the statistics computed from real data. We do exactly this when taking our model (with stochastic shocks) to the business cycle data.

6.4   Summary

The previous discussion about the linearization approach was very detailed, so it is worth summarizing the main steps involved.

1. Obtain the equations characterizing socially optimal (equivalently, equilibrium allocations)

2. Linearize them

3. Guess a linear policy function and solve for the undetermined coefficients of this policy function

4. Analyze the results by plotting the policy functions, impulse responses, simulations and compute statistics of the artificial data generated by the model. Finally compare them to real data.

5. Interpret the results.

This sounds very involved, but the nice thing is that a lot of these steps can be carried out by a computer, since they are purely mechanical. What is more, there now exists pre-programed software that does it for you, so you don't have to do it by yourself. The toolkit by Harald Uhlig that Philip is and will be using in the Tutorials is such freely available software. What still requires human work, however, are steps 1.,2. and 5. The painful step 2. of linearizing the equations will hopefully be somewhat automated soon, but so far you still have to do it by yourself.

Chapter 7

A Note on Economic Growth

So far the model we described does not display long-run growth, since the economy converges to its steady state. The data, in contrast, shows long-run growth at a positive rate. Part of this growth in the data is due to population growth, but even GDP per capita grows at a positive rate per capita. While long-run growth is not our main interest, it still would be nice if our model is consistent with the long-run growth observations as well. Fortunately this is quite easy to achieve.

7.1   Preliminary Assumptions and Definitions

Assume that the population and labor force grows at constant net growth rate n, so that the number of workers is given by

$$
\begin{aligned}
n_t &= (1+n)^t n_0 \\
&= (1+n)^t
\end{aligned}
$$

where $n_0 = 1$ is the size of the labor force at period 0. Furthermore assume that the production function is given by

$$
Y_t = AK_t^\alpha \left((1+g)^t n_t \right)^{1-\alpha}
$$

where g is the growth rate of technological, labor augmenting progress. Thus the aggregate resource constraint is given by

$$
C_t + K_{t+1} - (1-\delta)K_t = AK_t^\alpha \left((1+g)^t n_t \right)^{1-\alpha}
$$

I used capital letters for output and capital now, since now it is important to distinguish between aggregate and per capita variables. All aggregate variables will now be growing because of population growth and technical progress; thus

our aim is to reformulate the economy in terms of variables that are, at least potentially, constant over time.

Thus a few tedious, but useful definitions: let

$$
\begin{aligned}
c_t &= \frac{C_t}{(1+n)^t} \\
y_t &= \frac{Y_t}{(1+n)^t} \\
k_t &= \frac{K_t}{(1+n)^t}
\end{aligned}
$$

denote per capita variables and

$$
\begin{aligned}
\tilde{c}_t &= \frac{C_t}{(1+n)^t(1+g)^t} = \frac{c_t}{(1+g)^t} \\
\tilde{y}_t &= \frac{Y_t}{(1+n)^t(1+g)^t} = \frac{y_t}{(1+g)^t} \\
\tilde{k}_t &= \frac{K_t}{(1+n)^t(1+g)^t} = \frac{k_t}{(1+g)^t}
\end{aligned}
$$

denote variables per effective unit of labor.

The social planner problem is now given by

$$
\max_{\{c_t, k_{t+1}\}_{t=0}^T} \sum_{t=0}^T \beta^t u(c_t)
$$

subject to

$$
\begin{aligned}
C_t + K_{t+1} - (1-\delta)K_t &= AK_t^\alpha \left((1+g)^t n_t\right)^{1-\alpha} \\
c_t &\geq 0 \text{ and } K_0 > 0 \text{ given}
\end{aligned}
$$

7.2   Reformulation of Problem in Efficiency Units

We want to rewrite this problem in terms of variables that are not constantly growing over time, that is, in terms of the ˜ variables. First note that $K_0 = k_0 = \tilde{k}_0$, since $(1+n)^0 = (1+g)^0 = 1$. Second, $c_t \geq 0$ if and only if $\tilde{c}_t \geq 0$, so the only two things we have to worry about are the resource constraint and the utility function.

Divide the resource constraint by $(1+n)^t(1+g)^t$ to obtain

$$
\frac{C_t}{(1+n)^t(1+g)^t} + \frac{K_{t+1}}{(1+n)^t(1+g)^t} - \frac{(1-\delta)K_t}{(1+n)^t(1+g)^t} = \frac{AK_t^\alpha \left((1+g)^t n_t\right)^{1-\alpha}}{(1+n)^t(1+g)^t}
$$

and, using the variable definitions

$$
\tilde{c}_t + \frac{K_{t+1}}{(1+n)^t(1+g)^t} - (1-\delta)\tilde{k}_t = \frac{AK_t^\alpha \left((1+g)^t n_t\right)^{1-\alpha}}{(1+n)^t(1+g)^t}.
$$

But now note that

$$\frac{K_{t+1}}{(1+n)^t(1+g)^t} = \frac{K_{t+1}}{(1+n)^{t+1}(1+g)^{t+1}} * (1+g)(1+n) = (1+g)(1+n)\tilde{k}_{t+1}$$

$$\frac{AK_t^\alpha\left((1+g)^t n_t\right)^{1-\alpha}}{(1+n)^t(1+g)^t} = A\frac{K_t^\alpha}{[(1+n)^t(1+g)^t]^\alpha} * \frac{((1+g)^t n_t)^{1-\alpha}}{[(1+n)^t(1+g)^t]^{1-\alpha}} = A\tilde{k}_t^\alpha$$

and thus the resource constraint becomes

$$\tilde{c}_t + (1+g)(1+n)\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t = A\tilde{k}_t^\alpha.$$

Finally let us work on the lifetime utility function. Assume that the period utility function of the constant relative risk aversion form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where σ is a parameter.[1] If $\sigma = 1$, we take the utility function to be $u(c) = \log(c)$. Note that with this utility function we have that

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma} = \frac{(\tilde{c}_t(1+g)^t)^{1-\sigma}}{1-\sigma} = (1+g)^{t(1-\sigma)}\frac{\tilde{c}_t^{1-\sigma}}{1-\sigma}$$

Thus we can rewrite the lifetime utility function of the representative family as

$$\sum_{t=0}^{T}\beta^t u(c_t) = \sum_{t=0}^{T}\beta^t(1+g)^{t(1-\sigma)}u(\tilde{c}_t)$$

$$= \sum_{t=0}^{T}\tilde{\beta}^t u(\tilde{c}_t)$$

where $\tilde{\beta} = \beta(1+g)^{1-\sigma}$.

After all the smoke has settled, the social planner problem is given by

$$\max_{\{c_t,k_{t+1}\}_{t=0}^T}\sum_{t=0}^{T}\tilde{\beta}^t u(\tilde{c}_t)$$

subject to

$$\tilde{c}_t + (1+g)(1+n)\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t = A\tilde{k}_t^\alpha$$

$$\tilde{c}_t \geq 0 \text{ and } \tilde{k}_0 > 0 \text{ given}$$

---

[1]Note that the paramter σ is equal to the coefficient of relative risk aversion:

$$\frac{-cu''(c)}{u'(c)} = \sigma$$

and thus measures how risk averse the household is. The higher σ, the higher is the risk aversion of the household.

7.3   Analysis

Again forming the Lagrangian

$$
\begin{aligned}
\mathcal{L} \;=\;& \sum_{t=0}^{T} \tilde{\beta}^{t} u(\tilde{c}_t) + \sum_{t=0}^{T} \lambda_t \left(A\tilde{k}_t^{\alpha} - \tilde{c}_t - (1+g)(1+n)\tilde{k}_{t+1} + (1-\delta)\tilde{k}_t \right) \\
=\;& \ldots + \tilde{\beta}^{t} u(\tilde{c}_t) + \tilde{\beta}^{t+1} u(\tilde{c}_{t+1}) + \ldots \\
& + \lambda_t \left(A\tilde{k}_t^{\alpha} - \tilde{c}_t - (1+g)(1+n)\tilde{k}_{t+1} + (1-\delta)\tilde{k}_t \right) \\
& + \lambda_{t+1} \left(A\tilde{k}_{t+1}^{\alpha} - \tilde{c}_{t+1} - (1+g)(1+n)\tilde{k}_{t+2} + (1-\delta)\tilde{k}_{t+1} \right) + \ldots
\end{aligned}
$$

Taking first order conditions with respect to $\tilde{c}_t, \tilde{c}_{t+1}$ and $\tilde{k}_{t+1}$ yields

$$
\begin{aligned}
\tilde{\beta}^{t} u'(\tilde{c}_t) &= \lambda_t \\
\tilde{\beta}^{t+1} u'(\tilde{c}_{t+1}) &= \lambda_{t+1} \\
(1+g)(1+n)\lambda_t &= \lambda_{t+1} \left(\alpha A\tilde{k}_{t+1}^{\alpha-1} + (1-\delta) \right)
\end{aligned}
$$

Dividing the second equation by the first yields

$$
\frac{\lambda_{t+1}}{\lambda_t} = \frac{\tilde{\beta} u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)}
$$

From the third equation we have

$$
\frac{\lambda_{t+1}}{\lambda_t} = \frac{(1+g)(1+n)}{\left(\alpha A\tilde{k}_{t+1}^{\alpha-1} + (1-\delta) \right)}
$$

and thus combining the two yields

$$
\frac{\tilde{\beta} u'(\tilde{c}_{t+1})}{u'(\tilde{c}_t)} = \frac{(1+g)(1+n)}{\left(\alpha A\tilde{k}_{t+1}^{\alpha-1} + (1-\delta) \right)}
$$

$$
\tilde{\beta} u'(\tilde{c}_{t+1}) \frac{\left(\alpha A\tilde{k}_{t+1}^{\alpha-1} + (1-\delta) \right)}{(1+g)(1+n)} = u'(\tilde{c}_t) \tag{7.1}
$$

which is almost exactly the Euler equation we found when abstracting from population and productivity growth, with the exception that the marginal product of capital has to be divided by $(1+g)(1+n)$.

7.4   The Balanced Growth Path

We can repeat exactly the same analysis of a steady state in the model with growth. But if $\tilde{c}_t$ is constant over time, per capita consumption $c_t = \tilde{c}_t(1+g)^t$

grows at constant rate g; thus such a situation is not called a steady state, but a balanced growth path (since all per capita variables grow at constant rates). Now let us characterize the balanced growth path. Since $\tilde{c}_t = \tilde{c}_{t+1}$, the balanced growth path effective capital stock $\tilde{k}^*$ satisfies

$$\tilde{\beta} \frac{\left(\alpha A \tilde{k}^{\alpha-1} + (1-\delta) \right)}{(1+g)(1+n)} = 1$$

or

$$\frac{(1+g)^{1-\sigma} \left(\alpha A \tilde{k}^{\alpha-1} + (1-\delta) \right)}{(1+\rho)(1+g)(1+n)} = 1$$

$$\tilde{k}^* = \left(\frac{(1+\rho)(1+g)(1+n)(1+g)^{\sigma-1} - (1-\delta)}{\alpha A} \right)^{\frac{1}{\alpha-1}}$$

$$= \left(\frac{\alpha A}{(1+\rho)(1+g)(1+n)(1+g)^{\sigma-1} - (1-\delta)} \right)^{\frac{1}{1-\alpha}}$$

Note that if the period function is logarithmic, that is $\sigma = 1$, and the terms $\rho g, \rho n, gn$ and ρng are sufficiently small (which is the case as long as ρ, n, g are small), then

$$\tilde{k}^* = \left(\frac{\alpha A}{\rho + g + n + \delta)} \right)^{\frac{1}{1-\alpha}}$$

which is as before, only that the physical depreciation rate δ has to be augmented by the population growth rate and the growth rate of technological progress.

From the resource constraint effective consumption is given by

$$\begin{aligned} \tilde{c}^* &= A\tilde{k}_t^\alpha - (1+g)(1+n)\tilde{k}_{t+1} + (1-\delta)\tilde{k}_t \\ &= A\left(\tilde{k}^*\right)^\alpha - \left[(1+g)(1+n) - (1-\delta)\right]\tilde{k}^* \\ &\approx A\left(\tilde{k}^*\right)^\alpha - \left[g + n + \delta\right]\tilde{k}^* \end{aligned}$$

and effective output in the balanced growth path is given by

$$\tilde{y}^* = A \left(\tilde{k}^* \right)^\alpha$$

which in the log-utility case equals

$$y^* = A \left(\frac{\alpha A}{\rho + g + n + \delta} \right)^{\frac{\alpha}{1-\alpha}} = A^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\rho + g + n + \delta} \right)^{\frac{\alpha}{1-\alpha}} \qquad (7.2)$$

As discussed before, all per-capita variables grow at constant rate g in the balanced growth path, and all aggregate variables at rate $n + g$ (approximately, using $(1+n)(1+g) \approx 1 + n + g$). How about wages and interest rates, along the balanced growth path?

As before, the representative firm solves the maximization

$$\max AK_t^\alpha \left((1+g)^t n_t\right)^{1-\alpha} - w_t n_t - (r_t - \delta)K_t$$

where N_t and K_t is the number of workers and capital the firm hires. The first older conditions read as

$$w_t = (1-\alpha)(1+g)^t A \left(\frac{K_t}{(1+g)^t n_t}\right)^\alpha = (1-\alpha)(1+g)^t A \left(\tilde{k}_t\right)^\alpha$$

$$r_t = \alpha A \left(\frac{K_t}{(1+g)^t n_t}\right)^{\alpha-1} - \delta = \alpha A \left(\tilde{k}_t\right)^{\alpha-1} - \delta$$

Along the balanced growth path $\tilde{k}_t$ is constant, and thus the real interest rate r_t is constant and the real wage w_t is growing at a constant rate g.

To summarize, with economic growth it is as easy, or as hard, to solve this model that without growth. The presence of economic growth affects, however, the choice of the parameter values. How to pick these parameter values in discussed next.

Chapter 8

Calibration

Let us first collect all the parameters of the model. There are three sets of parameters we need to choose

- Technology parameters: (A, α, δ, g)

- Demographic parameters: (n)

- Preference Parameters: (ρ, σ)

The process of choosing these parameters is often called calibration. The idea is the following: we want to make the model's predictions to match certain observations from the data. Since we are interested in the business cycle properties of the model and do not want to cheat by choosing parameter values that help the model deliver good business cycle implications, we rather choose parameter values such that the *long run* implications of the model matches long run average observations from the data. Obviously the long run facts differ by countries. I will present a choice of parameter values calibrated to US data. It is interesting, and a good start for a diploma thesis, to do exactly the same analysis for other countries as well.

First we have to choose the length of a period, since our growth rates, for example, refer to growth rates from one period to the next. Therefore it is obviously crucial to specify how long a period lasts. Since most of business cycle research is done with quarterly data, we will use as period length a quarter.

8.1 Long Run Growth Rates

In the model n equals the growth rate of the population, which equals the growth rate of the labor force in the model as well. The annual average population growth rate in the US is about 1.1% per year. Thus the quarterly growth rate

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of the population (labor force) solves

$$(1+n)^4 = 1.011 \text{ or}$$
$$n = (1.011)^{\frac{1}{4}} - 1 \approx 0.27\%$$

We follow the same logic for the growth rate of technology g, which gives the growth rate of output per capita (per worker or per hour worked, all the same in the model). Between 1947 and 2004 the average growth rate of GDP per capita was 2.2% per year, and thus

$$(1+g)^4 = 1.022 \text{ or}$$
$$g = (1.022)^{\frac{1}{4}} - 1 \approx 0.55\%$$

8.2   Capital and Labor Share

We saw above that the production function parameter α equals the capital share in the model. Thus we want to choose a value for α that corresponds to the long run capital share in the data. At first this seems straightforward to do, until one realizes that it is not completely obvious how to compute the labor share and capital share in the data. One important problem is proprietor's income, that is, the income that people earn that own and run their business, supplying both their labor and their capital to the business. Another problem is the imputation of rental income for owner-occupied housing. If you own the house you live in, conceptually the rent that you should charge yourself for your home should count as capital income (since the rent a landlord gets from her tenants counts as capital income). In practice the statistics often do not capture this.

Without going into the dirty details we choose as compromise a $\alpha = \frac{1}{3}$, reflecting a long-run labor share in the data of roughly $\frac{2}{3}$. The range of estimates used in the literature (at least the one I am aware of) is $\alpha \in [0.25, 0.4]$.

8.3   The Depreciation Rate

Output is divided between consumption and investment. In the BGP investment equals $\tilde{i}^* = [g + n + \delta] \tilde{k}^*$. Thus the investment share of output equals

$$\frac{I}{Y} = \frac{\tilde{i}^*}{\tilde{y}^*} = (g + n + \delta)\frac{\tilde{k}^*}{\tilde{y}^*} = (g + n + \delta)\frac{K}{Y} \text{ and thus}$$
$$(g + n + \delta) = \frac{I/Y}{K/Y}$$
$$\delta = \frac{I/Y}{K/Y} - n - g$$

Thus with long-run averages for the investment-output ratio and the capital-output ratio from the data we can pin down δ. In US data the investment-output

share is roughly 25% in the long run, and the capital-output ratio is roughly 2.6 on an annual level. Note that the capital stock is a stock, and thus refers to a variable at a point in time. In contrast, output and investment are flow variables, and refer to a period of time (and thus depend on the period length). Thus if the annual capital-output ratio is 2.6, the quarterly ratio is $2.6*4 = 10.4$, reflecting the fact that quarterly GDP is one forth of annual GDP.

Thus we have as quarterly depreciation rate

$$\delta = \frac{0.25}{10.4} - 0.0027 - 0.0055 = 1.6\%$$

Here we also see the importance of incorporating or ignoring growth in the model: abstracting from growth we would have chosen a depreciation rate of $\delta = 2.42\%$ per quarter.

8.4 The Technology Constant A

The choice of A simply pins down the unit of measurement. Doubling A simply doubles all economic variables, without changing anything else. Often economists set $A = 1$ or set A such that steady state output $\tilde{y} = 1$ (conditional on all other parameters this is easily done using (7.2)). But it is a good check of your answers to questions (or your computer code) that indeed A is simply a normalization, and multiplying it by some number should multiply all economic variables by the same number (apart from ratios of variables and the interest rate, of course, which should remain unchanged).

8.5 Preference Parameters

The crucial equation for choosing preference parameters is the Euler equation (7.1)

$$\tilde{\beta} u'(\tilde{c}_{t+1}) \frac{\left(\alpha A \tilde{k}_{t+1}^{\alpha-1} + (1 - \delta)\right)}{(1 + g)(1 + n)} = u'(\tilde{c}_t)$$

which in the BGP becomes

$$\tilde{\beta} \frac{\left(\alpha A \tilde{k}_{t+1}^{\alpha-1} + (1 - \delta)\right)}{(1 + g)(1 + n)} = 1$$

Using the fact that the real interest rate equals the marginal product of capital net of depreciation, this becomes

$$\tilde{\beta} \frac{1 + r^*}{(1 + g)(1 + n)} = 1 \text{ or}$$

$$\frac{\beta(1 + g)^{1-\sigma}(1 + r^*)}{(1 + g)(1 + n)} = 1 \text{ or}$$

$$1 + r^* = (1 + g)(1 + n)(1 + \rho)(1 + g)^{\sigma-1}$$

Parameter	A	α	δ	σ	ρ
$g = 0, n = 0$	1	0.33	2.42%	1	1%
$g = 0.55\%, n = 0$	1	0.33	1.87%	1	0.45%
$g = 0.55\%, n = 0.27\%$	1	0.33	1.6%	1	0.2%

Table 8.1: Parameter Values

The parameter σ is hard to pin down using long-run observations, so economists often pick it independently. The preferred choice is $\sigma = 1$, that is logarithmic utility, in which case

$$1 + r^* = (1 + g)(1 + n)(1 + \rho)$$

Finally, the time discount rate is chosen such that, conditional on the choices for g and n, the balanced growth path of the model has an interest rate that matches its long run average in the data. Of course there are many real interest rates in the data, and the real interest rate in the model is both the risk-free real interest rate as well as the real return on capital. The former has an annual average of about 1%, the latter of about $7 - 8\%$. As a compromise we target an r^* of 4% per annum, or 1% quarterly. Conditional on the values of g, n chosen above this yields $\rho = 0.002 = 0.2\%$ quarterly. Again note that when calibrating an economy without growth, one would choose a $\rho = 1\%$ quarterly.

8.6   Summary

The following Table 8.1 summarizes the common choice of parameters for the model, both when calibrating a model without and with growth. The period length is a quarter.

Chapter 9

Adding Labor Supply

Now we could in principle simulate the economy, compute impulse responses or any other statistic that characterizes the equilibrium of the model. Before that we have to address two shortcomings, however. First, as we saw in the data section, the amount of labor used in production varies with the business cycle. In fact, apart from GDP (growth) itself, the unemployment rate is possibly the most important indicator of the business cycle, with high employment characterizing booms and low employment (high unemployment) characterizing recessions. Therefore in this section we now give households the opportunity to adjust their labor supply. Second, and this will be done in the next subsection we will introduce technology shocks that will make our economy to actually display business cycles. The endogenous response of employment to these shocks will then amplify the effects of these shocks on GDP.

In the standard real business cycle model the labor market is modelled as completely frictionless spot market where workers are paid their marginal product as wage. This wage clears the labor market, and thus at the market wage everyone that wants to work can do. There is no involuntary unemployment (whatever that means), but certainly households that do not find it preferable to work may choose not to supply any labor. Thus in no sense do we rule out unemployment (or better, non-employment).

9.1 The Modified Social Planner Problem

Lifetime preferences of the representative household become

$$\sum_{t=0}^{T} \beta^t \left[u(c_t) - \psi l_t \right]$$

where $l_t \in [0, 1]$ is the total number of hours the household works. We normalize the total time the household has available in a given period to 1. The number ψ is a parameter and determines how painful it is for a household to work,

with higher ψ implying higher disutility from work. The parameter ψ can also be interpreted as the marginal utility of leisure.[1] The social planner problem becomes[2]

$$\max_{\{c_t,k_{t+1},l_t\}_{t=0}^T} \sum_{t=0}^T \beta^t \left[u(c_t) - \psi l_t\right] \qquad (9.1)$$

$$\text{subject to}$$

$$c_t + k_{t+1} - (1-\delta)k_t = A k_t^\alpha l_t^{1-\alpha}$$

$$c_t \geq 0, \; l_t \in [0,1] \text{ and } k_0 > 0 \text{ given}$$

The specification of disutility of labor is peculiar, but, as we will see, necessary to get good business cycle results. It is linear in the disutility of labor, which will imply that labor supply reacts quite strongly to changes in wages. Since empirically labor input is quite volatile over the business cycle we will need every help we can get to make it volatile in the model as well. A strong response of labor supply to hours is therefore desirable.

9.2   Labor Lotteries

But how can we justify that the marginal disutility of labor is constant at ψ, whereas the marginal utility of consumption is strictly decreasing (remember that $u'' < 0$ was assumed)? Here is a trick that goes back to Richard Rogerson (*Journal of Monetary Economics*, 1988), one of Ed Prescott's most famous students. Suppose that the utility function is given by

$$\sum_{t=0}^T \beta^t \left[u(c_t) - v(l_t)\right]$$

where the function v satisfies $v' > 0$ and $v'' > 0$. That is, working more reduces utility, and the more you work the more hurts an additional hour of work. Assume that households can either work full time, $l_t = 1$ or not at all, $l_t = 0$,

---

[1] The restriction to at most one unit of work is a simple normalization. Suppose a household can supply at most h hours of work and the utility from leisure $h - l$ is given by

$$\psi(h-l)$$

then

$$\psi(h-l) = \psi h - \psi l.$$

But since ψh is simply a constant and adding constants to the utility function does not change the optimal labor (or consumption) choice, both formulation of the utility function give exactly the same outcomes.

[2] For simplicity we abstract from technology and population growth here. If the population is growing and there is technological progress we have exactly the same problem, but the production function would read as

$$AK_t^\alpha \left((1+g)^t n_t l_t\right)^{1-\alpha}$$

that is, they either have a job or they don't. Now assume that the social planner (or some unemployment agency) provides full insurance against being unemployed: no matter whether employed or unemployed, you receive the same amount of consumption. For this to work obviously it cannot be possible for agents to shirk unobservably, otherwise all people would claim to not be able to find a job (because they don't like to work and still get the same consumption even if they don't work). Finally denote by π_t the fraction of the population that the planner chooses to work (that is, π_t is a choice variable of the social planner) and by c_t the consumption level that both employed and unemployed get. Finally assume that all people get picked to work the same probability, so π_t is also the probability that a particular agent gets picked (remember we have a big number of identical agents of total number equal to 1). What the planner effectively does is to play a labor lottery with full consumption insurance.

Then expected utility in the current period is given as

$$
\begin{aligned}
E\left\{u(c_t) - v(l_t)\right\} &= \pi_t\left[u(c_t) - v(l_t = 1)\right] + (1 - \pi_t)\left[u(c_t) - v(l_t = 0)\right] \\
&= u(c_t) - \pi_t\left[v(1) - v(0)\right] - v(0)
\end{aligned}
$$

But note that since $v(1), v(0)$ are just two numbers and we can always ignore constants added to the utility function (they don't change first order conditions and thus optimal choices), we can rewrite the effective utility function as

$$
E\left\{u(c_t) - v(l_t)\right\} = u(c_t) - \psi\pi_t
$$

where $\psi = [v(1) - v(0)] > 0$ and we simply dropped the constant $-v(0)$. Replacing the name of the choice variable π_t by l_t gives back our original utility function. Also note that in the production function total labor input is equal to the fraction of workers working, π_t times their time worked $(= 1)$, so total labor input equals $\pi_t = l_t$. Thus the assumption of labor lotteries plus perfect consumption insurance justifies the usage of the particular functional form of disutility of labor in the utility function. Below we will describe how one can obtain the same result in a competitive market situation. But first we want to analyze the optimality conditions of the social planner problem with labor supply.

9.3 Analyzing the Model with Labor

Writing down the Lagrangian and ignoring the inequality constraints yields

$$
\begin{aligned}
L &= \sum_{t=0}^{T}\beta^t\left[u(c_t) - \psi l_t\right] + \sum_{t=0}^{T}\lambda_t\left[Ak_t^\alpha l_t^{1-\alpha} + (1 - \delta)k_t - c_t - k_{t+1}\right] \\
&= \dots\beta^t\left[u(c_t) - \psi l_t\right] + \lambda_t\left[Ak_t^\alpha l_t^{1-\alpha} + (1 - \delta)k_t - c_t - k_{t+1}\right] + \\
&\quad \beta^{t+1}\left[u(c_{t+1}) - \psi l_{t+1}\right] + \lambda_{t+1}\left[Ak_{t+1}^\alpha l_{t+1}^{1-\alpha} + (1 - \delta)k_{t+1} - c_{t+1} - k_{t+2}\right] + \dots
\end{aligned}
$$

Taking first order conditions with respect to $c_t, l_t, c_{t+1}, k_{t+1}$ and setting to zero yields

$$\beta^t u'(c_t) = \lambda_t \tag{9.2}$$

$$\lambda_t(1-\alpha)A\left(\frac{k_t}{l_t}\right)^\alpha = \beta^t\psi \tag{9.3}$$

$$\beta^{t+1}u'(c_{t+1}) = \lambda_{t+1} \tag{9.4}$$

$$\lambda_t = \lambda_{t+1}\left[\alpha A\left(\frac{l_{t+1}}{k_{t+1}}\right)^{1-\alpha} + (1-\delta)\right] \tag{9.5}$$

Combining equations $(9.2), (9.4)$ and (9.5) yields the familiar intertemporal Euler equation

$$u'(c_t) = \beta u'(c_{t+1})\left[\alpha A\left(\frac{l_{t+1}}{k_{t+1}}\right)^{1-\alpha} + (1-\delta)\right] \tag{9.6}$$

whereas combining (9.3) with (9.2) yields a new *intra*temporal optimality condition

$$(1-\alpha)A\left(\frac{k_t}{l_t}\right)^\alpha = \frac{\psi}{u'(c_t)}. \tag{9.7}$$

This condition has a nice interpretation. It states that the social planner chooses an optimal allocation such that the marginal product of labor (the left hand side of (9.7)) equals the marginal rate of substitution between leisure and consumption (the right hand side). To see that this optimality condition makes sense, observe that it exactly equates the costs and benefits of an extra hour of work. The cost, in terms of utility, of a marginal increase in labor, equals ψ. The benefit is to increase production and thus consumption by the marginal product of labor, and thus utility from consumption by

$$(1-\alpha)A\left(\frac{k_t}{l_t}\right)^\alpha u'(c_t).$$

In order to analyze this model one now would do exactly the same steps as in the model without labor-leisure choice:

1. Find the deterministic steady state

2. Log-Linearize the resource constraints and the optimality conditions around that steady state

3. Feed these equations into the software package of your choice, pick parameter values and let the computer figure out policy functions, make up impulse responses and simulations. The main difference to the case without labor is that now one has to solve for three policy functions

$$\hat{k}_{t+1} = s_k\hat{k}_t$$
$$\hat{c}_t = s_c\hat{k}_t$$
$$\hat{l}_t = s_l\hat{k}_t$$

instead of two as in the previous model (of course consumption can be deduced from the resource constraint in both cases, so effectively what is new is the policy function for labor).

Since steps 2 and 3 are tedious and conceptually (but not mechanically) simple, let us just simply determine the steady state. Since in the steady state $c_t = c_{t+1}$, from equation (9.6) we have

$$1 = \beta \left[\alpha A \left(\frac{l}{k} \right)^{1-\alpha} + (1 - \delta) \right]$$

which determines the steady state capital-labor ratio as

$$\frac{k}{l} = \left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} \tag{9.8}$$

Equation (9.7) then determines consumption implicitly as

$$\frac{\psi}{u'(c)} = (1 - \alpha) A \left(\frac{k}{l} \right)^{\alpha} = (1 - \alpha) A \left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{\alpha}{1-\alpha}}$$

Without an assumption on the utility function we cannot proceed further, obviously. But if we assume $u(c) = \log(c)$ we get

$$c = \frac{(1 - \alpha) A}{\psi} \left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{\alpha}{1-\alpha}} \tag{9.9}$$

Finally we employ the resource constraint to solve for (k, l). In the steady state the resource constraint reads as

$$\begin{aligned} c &= A k^{\alpha} l^{1-\alpha} - \delta k \\ &= k \left[A \left(\frac{k}{l} \right)^{\alpha - 1} - \delta \right] \end{aligned}$$

Using plugging in from equations (9.8) and (9.9) we get

$$\frac{(1 - \alpha) A}{\psi} \left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{\alpha}{1-\alpha}} = k \left[\frac{\delta + \rho}{\alpha} - \delta \right]$$

or

$$k = \frac{\frac{(1-\alpha) A}{\psi} \left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{\alpha}{1-\alpha}}}{\left[\frac{(1-\alpha)\delta + \rho}{\alpha} \right]} = \frac{\alpha (1 - \alpha) A}{\psi \left[(1 - \alpha)\delta + \rho \right]} \left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{\alpha}{1-\alpha}}$$

which looks like a big mess, but still allows for nice interpretation. In particular, the steady state capital stock is increasing in the technology constant A and decreasing in the depreciation rate δ and the time discount rate ρ as well as the disutility of labor parameter ψ.

Finally, from (9.8) steady state hours worked are given by

$$
\begin{aligned}
l &= \left(\frac{\alpha A}{\delta + \rho}\right)^{\frac{-1}{1-\alpha}} k \\
&= \frac{\alpha(1-\alpha)A}{\psi\left[(1-\alpha)\delta + \rho\right]} * \frac{\delta + \rho}{\alpha A} = \frac{(1-\alpha)(\delta + \rho)}{\psi\left[(1-\alpha)\delta + \rho\right]} = \\
&= \frac{(1-\alpha)(\delta + \rho)}{\psi\left[(1-\alpha)(\delta + \rho) + \alpha\rho\right]} = \frac{1}{\psi\left[1 + \frac{\alpha\rho}{(1-\alpha)(\delta+\rho)}\right]} \\
&= \frac{1}{\psi\left[1 + \frac{\alpha}{(1-\alpha)\left[\frac{\delta}{\rho}+1\right]}\right]}
\end{aligned} \tag{9.10}
$$

Therefore the seemingly redundant rewritings were successful: we can unambiguously conclude that steady state labor supply is decreasing in ψ and α, increasing in the depreciation rate δ and decreasing in impatience ρ. Importantly, the optimal amount of work is independent of the technology level parameter A. This somewhat surprising fact is the result of two opposing effects that exactly cancel out in the case in which the utility function for consumption is logarithmic and disutility of labor is linear. A bigger A makes labor more productive, thus effectively leisure more expensive (because its opportunity costs increase). Thus it is optimal for the planner to substitute leisure for consumption and let the agent eat more and work more. But on the other hand there is an income effect: a higher A makes the economy generate the more output with the same inputs. As a consequence it is optimal to increase both consumption and leisure. Thus while consumption increases unambiguously (both income and substitution effect are positive), for leisure a negative substitution effect stands against a positive income effect, which in our utility specification they exactly cancel out. This will not be true for other utility functions, however.

9.4   A Note on Calibration

How should we pick ψ? Using the same principles as before we want our economy to reproduce a total amount of work in the model equal to the long run average in the data. Empirically people, on average, work about one third of their non-sleeping time, so we want to choose parameters such that l in equation (9.10) equals to 1/3. Thus

$$
\begin{aligned}
\frac{1}{3} &= \frac{1}{\psi\left[1 + \frac{\alpha}{(1-\alpha)\left[\frac{\delta}{\rho}+1\right]}\right]} \\
\psi &= \frac{3}{\left[1 + \frac{\alpha}{(1-\alpha)\left[\frac{\delta}{\rho}+1\right]}\right]}
\end{aligned}
$$

Conditional on the other parameter choices discussed in the calibration section (without growth), a number of $\psi = 2.62$ is necessary for this.

9.5 Intertemporal Substitution of Labor Supply: A Simple Example

In the previous section we showed that, under certain assumptions, steady state labor supply does not depend on the productivity parameter A. Now we want to demonstrate how agents respond to *temporarily* high labor productivity (equal to wages in the competitive equilibrium). We will show that agents will find it optimal to intertemporally substitute labor supply and work hard when they are productive, and work less hard when they are not. The extent to which this intertemporal substitution of labor supply occurs depends crucially on the form the disutility function for labor takes, which controls what is called the labor supply elasticity.

We demonstrate this in a simple example where agents live for two periods, don't discount the future and only value consumption in the second period. Furthermore we abstract from capital and capital accumulation, but let households or the social planner store output between the first and the second period. The main mechanism described here, however, will also be present in our general model.

Let A_1 denote labor productivity in the first period and A_2 denote labor productivity in the second period. The social planner problem becomes

$$\max \ln(c_2) - \psi l_1 - \psi l_2$$
$$\text{s.t. } c_2 = A_1 l_1 + A_2 l_2$$

Let us suppose that the constraints $l_1 \leq 1$ and $l_2 \leq 1$ are never binding. One can show that as long as $\psi > 1$ this assumption is satisfied. What does optimal labor supply in both periods look like? If $A_1 > A_2$, then the agent should work only in period 1, and if $A_2 > A_1$ she should only work in period 2. This is easy to see: an extra unit of work brings about an extra disutility of work of ψ, no matter when it is done and how much the agent already works. Thus she should always work that extra unit in the period in which she is more productive.

For concreteness, suppose that $A_1 > A_2$. Then $l_2 = 0$. Solving

$$\max \ln(c_2) - \psi l_1$$
$$\text{s.t. } c_2 = A_1 l_1$$

yields $l_1 = \frac{1}{\psi}$ and $c_2 = \frac{A_1}{\psi}$. In general the optimal solution to the Planner problem is given by the following table below. In the knife-edge case in which $A_1 = A_2$ the agent is indifferent between working in the first or second period, as long as total labor supply adds up to $\frac{1}{\psi}$. The key observation is the following: suppose that A_2 is the normal, long run average labor productivity. Then we see that when current productivity A_1 is higher than normal, the agent works more and when it is lower she works less.

Case	l_1	l_2	c_2
$A_1 > A_2$	$\frac{1}{\psi}$	0	$\frac{A_1}{\psi}$
$A_1 < A_2$	0	$\frac{1}{\psi}$	$\frac{A_2}{\psi}$
$A_1 = A_2 = A$	$\in [0, \frac{1}{\psi}]$	$\in [0, \frac{1}{\psi}]$	$\frac{A}{\psi}$

Table 9.1: Labor and Consumption Allocations

For the utility specification we chose the effects on labor supply of changes in productivity are very strong: even the slightest deviation from normal productivity results in large changes in labor supply. While this is partially due to the fact that we abstracted from capital and consumption in the first period, it also crucially depends on the fact that disutility of labor is linear. To see this, suppose instead that the utility function is given by

$$\ln(c_2) + \psi \ln(1 - l_1) + \psi \ln(1 - l_2)$$

where $1 - l_1$ is leisure in the first period and $1 - l_2$ is leisure in the second period. Maximizing this utility function subject to the resource constraint

$$c_2 = A_1 l_1 + A_2 l_2$$

yields as one condition

$$\frac{A_2}{A_1} = \frac{1 - l_1}{1 - l_2}$$

While the overall allocation is tedious to solve for, we can readily make two important observations. First, there is again intertemporal substitution of labor supply: if $A_1 > A_2$, then $l_1 > l_2$, that is, the agent responds to temporarily higher productivity by working more. Second, for this preference specification the household does not respond as drastically to differences in A_1 versus A_2. As long as the ratio $\frac{A_2}{A_1}$ is not too big, she works in both periods, and small changes in $\frac{A_2}{A_1}$ do not lead to drastic labor supply responses. Since in the data labor input varies substantially over the business cycle while real wages (labor productivity) only moderately so, it is not surprising that the linear disutility formulation has enjoyed bigger success.

We now introduce shocks to our production function constant A_t, from now on called Total Factor Productivity (TFP). An important factor of making output more volatile than the shocks to A_t fed into the model is the intertemporal substitution of labor supply just described. Before doing so, let us make a quick remark how the solution to the social planner problem translate into a competitive equilibrium.

9.6   A Remark on Decentralization

So far we have discussed the social planner problem and characterized socially optimal allocations. "Decentralization" refers to the question whether we can

make this socially optimal allocation into a competitive equilibrium with the right choice of prices. Here I will constrain myself to asserting that the welfare theorems still apply and thus we know that we can find prices that decentralize the socially optimal allocation. In fact we know what these prices are

$$
\begin{aligned}
r_t &= \alpha A \left(\frac{l_t}{k_t}\right)^{1-\alpha} - \delta \\
w_t &= (1-\alpha) A \left(\frac{k_t}{l_t}\right)^{\alpha}
\end{aligned}
$$

If with these prices, the consumer maximizes her lifetime utility, subject to the budget constraint

$$
c_t + k_{t+1} = w_t l_t + (1 + r_t) k_t
$$

she would choose exactly the same allocation $\{c_t, k_{t+1}, l_t\}_{t=0}^{T}$ as the social planner.[3]

---

[3] How would one decentralize these labor lotteries? Now indeed agents indeed would choose a lottery, i.e. a probability π_t of working. In addition all agents in the economy would write insurance contracts with each other. Those agents who then get to work would earn as labor income $w_t * 1$, but would only keep $\pi_t w_t$ and pay the rest to those people that don't get a job. Their labor income, inluding the transfer from those that work, amounts to exactly $\pi_t w_t$ as well. Why? There are $1 - \pi_t$ unemployed and π_t employed people. So transfers t have to solve

$$
\begin{aligned}
(1 - \pi_t)t &= \pi_t(1 - \pi_t)w_t \\
t &= \pi_t w_t.
\end{aligned}
$$

Thus such a scheme perfectly insures the people that don't work. Obviously the viability of such a transfer scheme depends crucially on the absence of so-called moral hazard problems. If people with a job could credibly misrepresent their situation and pretend not to have a job, they have every incentive to do so, because then they would not have to work and still receive the same consumption, on account of the insurance. Thus we need to, implicitly, assume that either people are honest or can be monitored at no cost.

Chapter 10

Stochastic Technology Shocks: The Full RBC Model

So far our economy does not display any cyclical fluctuations. Now we introduce exogenous shocks to the production technology parameter A_t, which we now allow to vary over time (over and above deterministic growth in productivity considered above).

10.1   The Basic Idea

The basic idea is the following. Suppose that in every period A_t can take the value A_l or A_h, and does so with probability $\frac{1}{2}$. Thus the average productivity equals $\bar{A} = \frac{1}{2}\left(A_l + A_h\right)$. Remember that our production function takes the form (again for simplicity we abstract from economic growth for the exposition)

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}.$$

Thus if current TFP is high, output will be high even if all production inputs stay the same. In addition we saw in the previous chapter that households (or the social planner) will optimally respond to a temporarily higher TFP by working harder, increasing production even further. The reverse logic applies to a negative productivity shock, $A_t = A_l$. The economy starts to display fluctuations that look like business cycles, driven by shocks to TFP A_t and further propagated by the endogenous response of labor supply. This is the basic idea of real business cycle theory.

There is still one obvious shortcoming. In the data we saw that business cycles were very persistent: good times were more likely followed by further good times than by bad times. With technology shocks where the probability of good and bad shock is independent of its past realization the model has a hard

time generating persistence. Why? Because after a good shock $A_t = A_h$ today tomorrow's shock is as likely to be good as bad. Labor supply simply responds to the current technology, so will not generate higher output tomorrow either. And finally, what is the dynamics of the capital stock? Investing in the capital stock today (i.e. increasing investment) only results in a higher capital stock to be used in production *tomorrow*. But tomorrow A_{t+1} is as likely to be high as it is low, and therefore the social planner (or private households) have no particularly strong motive to invest if today's shock is high. Thus there is no persistent effect on output due to higher capital accumulation either.[1]

Thus researchers in the RBC literature specify the TFP shocks not as independent over time (the probability of $A_{t+1} = A_h$ does not depend on the realization of A_t), but rather as a positively correlated process. If $A_t = A_h$ then the probability of $A_{t+1} = A_h$ is higher than if $A_t = A_l$. With this formulation one gets persistence of business cycles via two channels

- Output tomorrow will be higher if today's shock is good because tomorrow's shock is likely to be good as well.

- Looking forward, the planner today knows that if today's shock is high it is good to invest in the capital stock today because the investment today results in a higher capital stock tomorrow and that capital stock tomorrow is very productive because TFP is likely to be high tomorrow.

There is one drawback, however, from the formulation of technology shocks as persistent. We argued above that the labor supply response to a temporarily favorable TFP shock amplifies the business cycle in the model. But the current labor supply response will be smaller if the TFP shock is persistent because with a positive TFP shock today it is not only a good time to work harder today, but also very likely to be a good time to work tomorrow as well. Thus the increase in labor supply is smaller compared to a situation where productivity is as likely to be low or high tomorrow if is high today.

10.2   Specifying a Process for Technology Shocks

In the light of the previous discussion researchers specify the process for TFP as a persistent process. It is also common to allow for more than just two possible realizations of the technology shock; that is, instead of $A_t \in \{A_l, A_h\}$ an entire range of A_t is permitted.

Concretely, with the production function given as

$$y_t = A_t k_t^\alpha l_t^{1-\alpha}$$

---

[1]There may be a small income effect: higher TFP today increases output and part of that increased output will be used for higher future consumption. The way to do this in this model is to increase saving via higher capital accumulation, creating a little bit of persistence. But as we will see below this effect is quantitatively insufficient to generate business cycles of sufficient persistence..

we specify

$$A_t = Ae^{z_t} \text{ with} \tag{10.1}$$
$$z_t = \rho_z z_{t-1} + \varepsilon_t \tag{10.2}$$

where ε_t is a random shock, drawn in every period from the same normal distribution with zero mean and variance σ_z^2. Shocks in different periods are assumed to be independent, so that, technically speaking, $\{\varepsilon_t\}_{t=0}^{\infty}$ is a sequence of *iid* normally distributed random variables. The number ρ_z is a parameter that measures how persistent the technology shock is, that is, how important the past productivity shock for determining how big it is today. Finally A is the average productivity level.

At first this formulation seems puzzling, but simply take logs of (10.1) to obtain

$$\log(A_t) = \log(A) + z_t$$
$$\log(A_t) - \log(A) = z_t$$

That is, z_t is the (log-)deviation of the actual productivity level from its average.

Before analyzing the model with technology shocks we first want to collect some important properties of the process (10.2). From basic time series econometrics we recall that this process is called an autoregressive process of order 1, because the value of z_t only depends on the value in the last period, z_{t-1} and an *iid* shock. We assume that the process starts off with $z_{-1} = 0$; alternatively we could assume it started in the infinite past, in which case the initial value does not matter.

We are interested in the expectation and the variance of z_t. There are two different ways to measure this, depending on the time at which expectation and variance are taken. We call the unconditional expectation of z_t the expectation of z_t at time 0. The only thing we have observed at that point is $z_{-1} = 0$. By $E_0(z_t)$ denote this unconditional expectation. Similarly denote by

$$Var_0(z_t) = E_0\left[(z_t - E_0(z_t))^2\right]$$

the unconditional variance of z_t. More important for our purposes are the conditional expectation and variance, conditional on information available at the end of period $t - 1$. We denote these by $E_{t-1}(z_t)$ and $Var_{t-1}(z_t)$. The conditional expectations and variance are easy to derive. Remember that at the end of period $t - 1$ we know z_{t-1} and thus

$$E_{t-1}(z_t) = E_{t-1}[\rho_z z_{t-1} + \varepsilon_t]$$
$$= E_{t-1}[\rho_z z_{t-1}] + E_{t-1}[\varepsilon_t]$$
$$= \rho_z z_{t-1} + 0 = \rho_z z_{t-1}$$

since the shock ε_t has a zero mean. In addition

$$Var_{t-1}(z_t) = Var_{t-1}[\rho_z z_{t-1}] + Var_{t-1}(\varepsilon_t)$$
$$= \sigma_z^2$$

since at the end of period $t - 1$ the term $\rho_z z_{t-1}$ is known and thus has zero variance.

From these results we observe the following: when the households (or the social planner) make decisions in period $t - 1$ after having observed z_{t-1} they need to form expectations about TFP in period t, because this is important for them for deciding how much to invest and to work, since z_t determines how productive their labor and capital will be in period t. Now we see the importance of the persistence parameter ρ_z. If $\rho_z = 0$, then the best estimate of z_t at the end of period $t-1$ is 0 no matter what TFP is in period $t-1$. This was the case discussed in the previous section. If, on the other hand ρ_z is close to one, the best guess of productivity for period t is that it remains at the level of the previous period. Then, if TFP is high today it is a great time to invest since the newly purchased capital will likely be very productive tomorrow. The parameter σ_z measures how risky TFP is and thus may control the extent to which households or the social planner want to accumulate capital for precautionary reasons. But since our linearization techniques cannot capture this effect I do not want to discuss it further. The size of σ_z will also inform us how big the technology shocks are that we will feed into the model when we simulate it.[2]

10.3 Analysis

We now want to analyze the model with technology shocks. We will be a bit casual when dealing with the stochastic shocks and the expectation operator dealing with these shocks, since doing this very formally would require a host of additional notation. The final outcome of our analysis is however, perfectly correct, subject to the usual approximation error one makes when employing linearization techniques.

Again we will restrict attention to the social planner problem since the welfare theorems go through completely unchanged and we can easily make the social planner solution into a competitive equilibrium. The social planners prob-

---

[2]It is somewhat more tedious to compute the unconditional expectation and variance of z_t, but we find

$$E_0(z_t) = 0$$

$$Var_0(z_t) = \sigma^2 \sum_{\tau=0}^{t}$$

and finally, if the economy starts at $t = -\infty$

$$Corr(z_t, z_{t-1}) = \frac{Cov(z_t, z_{t-1})}{Std(z_t)Std(z_{t-1})} = \rho_z$$

and thus we see that also formally ρ_z measures the correlation between z_t and z_{t-1}.

lem reads as

$$\max_{\{c_t,k_{t+1},l_t\}_{t=0}^T} E_0 \sum_{t=0}^{T} \beta^t \left[u(c_t) - \psi l_t \right]$$

$$\text{subject to}$$

$$c_t + k_{t+1} - (1-\delta)k_t = Ae^{z_t}k_t^{\alpha}l_t^{1-\alpha}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_t$$

$$c_t \geq 0, \ l_t \in [0,1] \text{ and } k_0 > 0 \text{ given}$$

This problem is almost identical to the one without technology shocks. The only difference is that now the production function is subject to technology shocks whose stochastic process needs to be specified, and that now *expected* lifetime utility is being maximized. Note that the resource constraint has to hold under every possible realization of the TFP shock, not just in expectation.

The first order conditions look almost exactly like the ones without uncertainty. In particular, the intra- and intertemporal optimality conditions become

$$(1-\alpha)Ae^{z_t}\left(\frac{k_t}{l_t}\right)^{\alpha} = \frac{\psi}{u'(c_t)}$$

and

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \left[\alpha A e^{z_{t+1}} \left(\frac{l_{t+1}}{k_{t+1}} \right)^{1-\alpha} + (1-\delta) \right] \right\}$$

where E_t denotes the conditional expectation.[3] The key difference to the case without technology shocks is that now the Euler equation contains an expectation, since at time t when the decision about k_{t+1} is made the shock z_{t+1} is not yet known and the social planner has to form expectations about it when choosing k_{t+1}. Expectations are rational in the sense that the stochastic process the social planner (or the households) perceives coincides exactly with the true stochastic process governing z_t. Also note from the resource constraint that since output y_{t+1} is random (because z_{t+1} is random) consumption c_{t+1} is random as well, so the expectation is not only to be taken with respect to z_{t+1}, but also with respect to c_{t+1} (and also with respect to l_{t+1}, which is stochastic as well).

The procedure to solve for the optimal policy functions is the same as in the nonstochastic case: log-linearize the intra- and intertemporal optimality conditions around its steady state (with the shocks $z_t = 0$; this is called the deterministic steady state and coincides with the steady state of the deterministic model), the resource constraint and the equation governing the z_t-process, stick it into the software package and let the program solve for policy functions. There is one crucial difference to the nonstochastic case. In that case the current state of the economy was completely determined by the current capital stock,

---

[3]It would lead us too far astray to explicitly derive the intertemporal Euler equations in the case of uncertainty, so here I ask you to let me proceed on faith.

but now one also needs to know the current shock z_t, because the shock is a crucial part in determining current output. Thus the policy functions take the form

$$
\begin{aligned}
\hat{k}_{t+1} &= \gamma_k z_t + s_k \hat{k}_t \\
\hat{c}_t &= \gamma_c z_t + s_c \hat{k}_t \\
\hat{l}_t &= \gamma_l z_t + s_l \hat{k}_t
\end{aligned}
$$

where $(\gamma_k, \gamma_c, \gamma_l)$ and (s_k, s_c, s_l) are the numbers the software is determining for you. Note that the steady state value of z_t is $z^* = 0$ and thus we write the policy functions as functions of z_t and not $\hat{z}_t$.

In order to determine the optimal policy functions and simulate the model we need to pick the parameter values governing the process for z_t, that is, we need to choose ρ_z and σ_z. Since we want to ask whether our model can generate business cycles of realistic magnitude for empirically reasonable choices of ρ_z, σ_z, we cannot just make them out according to our liking, but rather want to determine them from the data. The procedure for doing so is described next.

10.4   What are these Technology Shocks and How to Measure Them?

Output in the model is produced according to the production function

$$
y_t = A_t k_t^\alpha \left[(1+g)^t l_t \right]^{1-\alpha}
$$

Taking logs we find

$$
\begin{aligned}
\log(y_t) &= \log(A_t) + \alpha \log(k_t) + (1-\alpha) \log(l_t) + (1-\alpha)t \log(1+g) \\
\log(A_t) &= \log(y_t) - \alpha \log(k_t) - (1-\alpha) \log(l_t) - (1-\alpha)t \log(1+g)
\end{aligned}
$$

Since we can measure y_t, l_t, k_t and g (the long run growth rate of GDP per capita) from the data (measuring the real capital stock is not entirely straightforward), we can, conditional on having picked a value for α, construct a time series for $\log(A_t)$. The variable A_t or $\log(A_t)$ so determined from the data is called the Solow residual and measures our infamous technology shocks in the data.

Now remember that

$$
\begin{aligned}
&\log(A_t) - \log(A) \\
&= z_t = \rho_z z_{t-1} + \varepsilon_t \\
&= \rho_z \left(\log(A_{t-1}) - \log(A) \right) + \varepsilon_t
\end{aligned}
$$

and thus

$$
\log(A_t) = (1 - \rho_z) \log(A) + \rho_z \log(A_{t-1}) + \varepsilon_t
$$

Thus with data on $\{\log(A_t)\}$ we can run the regression

$$
\log(A_t) = \alpha_1 + \alpha_2 \log(A_{t-1}) + \varepsilon_t
$$

and take the OLS estimate $\hat{\alpha}_2$ as our estimate for ρ_z. Denoting the regression residuals as

$$\hat{\varepsilon}_t = \log(A_t) - \hat{\alpha}_1 - \hat{\alpha}_2 \log(A_{t-1})$$

we obtain as estimate for σ_z^2

$$\hat{\sigma}_z^2 = \frac{1}{T} \sum_{t=0}^{T} \hat{\varepsilon}_t^2$$

Note that as before the constant A is simply a normalization of units. If one carries out this exercise with American quarterly data one obtains $\rho_z = 0.95$ and $\sigma_z = 0.007$.

Now we are ready to deduce the quantitative business cycle properties of our model. Before doing this in the results section below, we first want to discuss what these technology shocks could represent in the real world? Everything that effects the productivity of inputs in our economy. Positive shocks could represent the advent of new ideas of production, new technologies, good weather and so forth, negative shocks could represent oil price shocks, terrorist attacks and many other things that reduce total factor productivity below its long run growth trend (as given by the term $(1+g)^{t(1-\alpha)}$).

Part III

Evaluating the Model

In the previous part we collected all the necessary ingredients to use our model for business cycle analysis. In this part we will derive the quantitative properties of the model, with the use of computational techniques. We first derive and describe the basic quantitative properties of the model. We then try to disentangle how much of the business cycles resulting from this model is due to the exogenous shocks hitting the technology, and how much of is due to the endogenous response of households and firms (adjusting their labor supply and investment decisions). Once we have a satisfactory model of business cycles we ask to important applied questions. First, how costly are business cycles, that is, how much would citizens of a society gain if they could get rid off business cycles. This question is silent about how one possibly would achieve getting rid of business cycles. The last part of these notes then explores to what extent economic policy (monetary and fiscal policy) is suited to reduce or abolish business cycles, and whether it is in fact desirable to do so.

Chapter 11

Technology Shocks and Business Cycles

In this chapter we will document the basic business cycle properties of the RBC model, calibrated in the way described above. We will use two basic ways to summarize the predictions of the model, impulse response functions and summary statistics from simulations of the model.

11.1   Impulse Response Functions

The state variables of the Hansen model are the capital stock at the beginning of the period, k_t, and the current technology shock z_t. We now show how a shock to the current capital stock and the current technology level affect the optimal choices of the social planner (or equivalently, private households and firms). The thought experiment is the following: suppose the economy is at its deterministic steady state and all of a sudden the current capital stock increases by 1% in period 0. It is assumed that current and future technology shocks are at their mean, that is, $z_t = 0$ for all t. The impulse response functions trace out how the endogenous variables of the model (that is, capital in future periods, labor, consumption, output and investment) respond to the shock to the capital stock the economy comes into the current period with. We first observe from Figure 11.1 that in period 0, by construction, only the capital stock goes up, by 1%. In response investment strongly declines, because the capital stock is higher than optimal (note that we shock the capital stock, and do not assert that the increase in the capital stock is in any sense optimal). The decline in current investment leads, over time, to a reduction of the capital stock back to the deterministic steady state. The exogenous increase in the capital stock makes the economy wealthier. In response current consumption and leisure increase. However, it is optimal not to consume this additional output entirely in the current period, but rather to smooth the consumption increases over time. Thus consumption and leisure are not only higher in the period after the increase in the capital
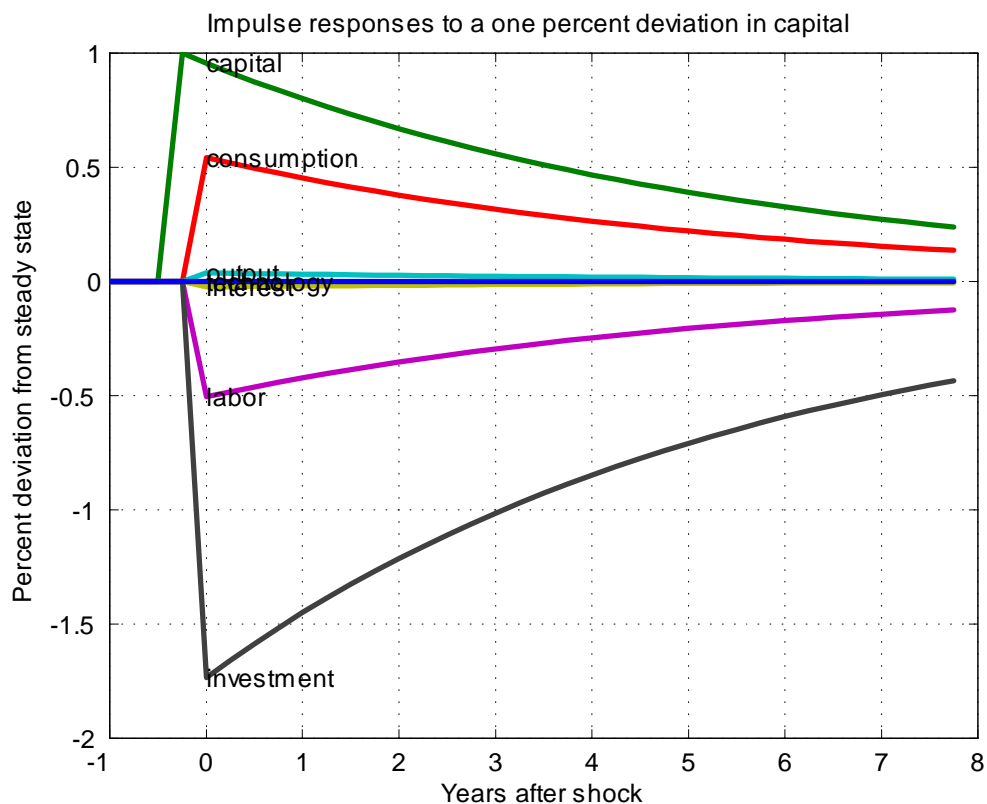
Figure 11.1: Impulse Response Function for the Hansen Model, Shock to Capital

stock, but persistently higher, and only come back slowly to the steady state over time. Finally, output slightly increases due to the increase in the capital stock. This effect is mitigated by the decline in labor supply, making the overall output response modest.

More important for understanding the business cycle properties of the model is the response of the model to a technology shock, the source of business cycles in our model. Again we assume that before period 0 the economy is in the deterministic steady state, and then at period 0 there is a positive technology shock, in the size of one standard deviation, that is $z_0 = \sigma_z$. After this shock the technology by assumption is not hit by further shocks, and thus the technology level follows the process

$$z_t = \rho z_{t-1}.$$

We see from Figure 11.2 that the technology shock jumps up in period 0, and then geometrically declines back to its steady state level $z = 0$. This is com-
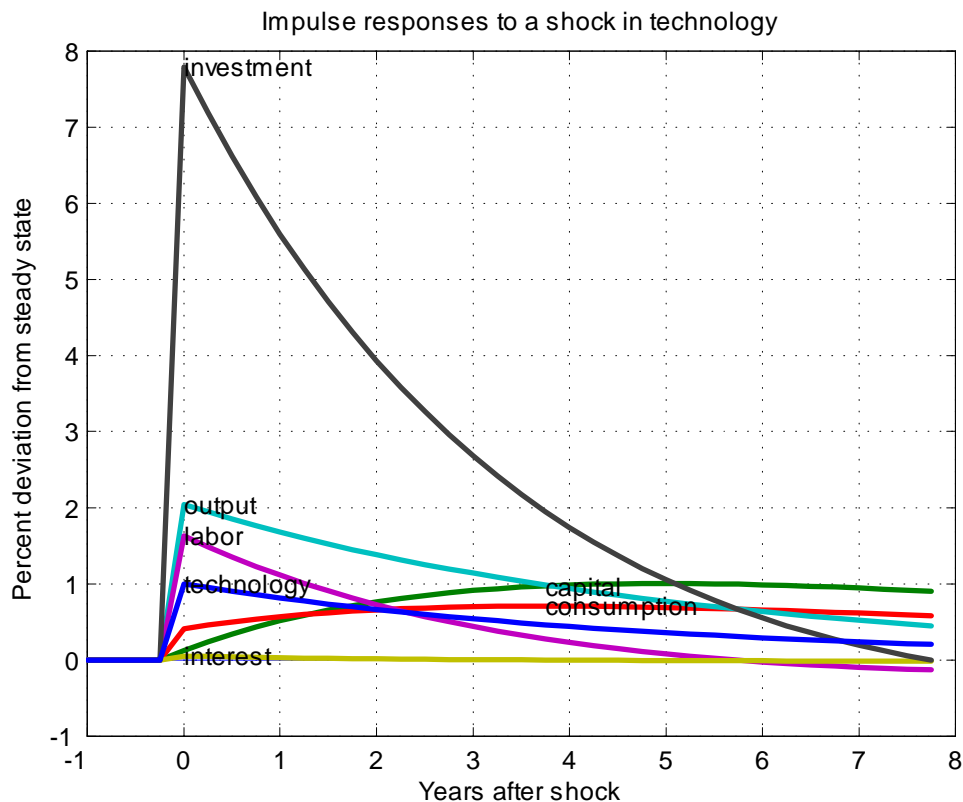
Figure 11.2: Implulse Response for the Hansen Model, Technology Shock

pletely by construction, and there is nothing surprising or endogenous about this. The key question is how the endogenous variables of model, capital, consumption, consumption etc. respond to the technology shock.

The most significant response is in private investment, which on impact increases by almost 8% on impact. An increase in technology makes capital more productive in the future, since future technology is expected to be higher (note that ρ is close to 1). The social planner responds optimally by immediately building up the capital stock. Labor supply also responds positively to the increase in productivity, albeit not as strongly as investment. Consequently output increases by more than the technology shock: this is the intertemporal substitution of labor supply. Consumption also increases, albeit very little, on impact. The small increase in consumption is due to the fact that it is optimal to devote a lot of the extra production to investment to reap the benefits of higher productivity in the future. Over time investment declines back to the steady state quite quickly, but consumption remains high for a much longer time, fueled

by the now higher capital stock and the higher productivity. Eventually the technology level gets back to the steady state, and so do output, consumption, the capital stock and labor.

The key observations from the impulse response analysis for business cycles is the very strong response of investment on impact of the technology shock, the positive labor supply response, the persistent effect on output due to the persistence of the technology shock and the increase of the capital stock. These properties of the policy functions will translate into the business cycle properties of the model. These properties we will now document by simulating the model and then comparing the simulated time series from the model to the business cycle statistics from the data.

11.2   Comparing Business Cycle Statistics of Model and Data

Once we have the policy functions we can easily simulate the model. Start with an initial condition for the capital stock, say the steady state level, so that $\hat{k}_0 = 0$, and with an initial condition for z, say $z_0 = 0$. Then draw a sequence of technology shocks[1] $\{\varepsilon_t\}_{t=0}^T$ and construct $\{z_t\}_{t=1}^T$ from the equation

$$z_t = \rho_z z_{t-1} + \varepsilon_t$$

Once we have the initial condition $\hat{k}_0 = 0$ and the technology shocks z_t, we can use the policy functions delivered from the software

$$\begin{aligned}
\hat{k}_{t+1} &= \gamma_k z_t + s_k \hat{k}_t \\
\hat{c}_t &= \gamma_c z_t + s_c \hat{k}_t \\
\hat{l}_t &= \gamma_l z_t + s_l \hat{k}_t
\end{aligned}$$

to determine time series $\{\hat{k}_{t+1}, \hat{c}_t, \hat{l}_t\}_{t=1}^T$ and thus of course time series for $\{k_{t+1}, c_t, l_t\}_{t=1}^T$. Now that we artificial data from the model, we can compute exactly the same statistics as we did form the real data. The model is deemed to be a good model of business cycles if the statistics from the model match up favorably with those from the data.

In table 11.1 we summarize the basic business cycle properties of the model, and, as comparison, of the data, from table 2.1.

We observe that the model is able to generate output volatility of the same magnitude as found in the data. In addition, the persistence of business cycles in the model is comparable to that in the data, although the persistence in the model is somewhat lower than that in the data. Figure 11.3 shows a simulation

---

[1]Drawing a seqeunce of independent, normally distributed random variables is a fairly standard problem for which reliable software exists. The resulting numbers are however, only pseudo-random numbers, that is, they are generated according to a deterministic function, but look like random numbers. If you are interested in the mathematical details, I have further references for you.

Variable	Mean	St. Dv.	A(1)	A(2)	A(3)	A(4)	A(5)
Data	0%	1.7%	0, 84	0, 60	0, 32	0, 08	−0, 10
Model	0%	2.1%	0, 72	0, 42	0, 16	−0, 05	−0, 21

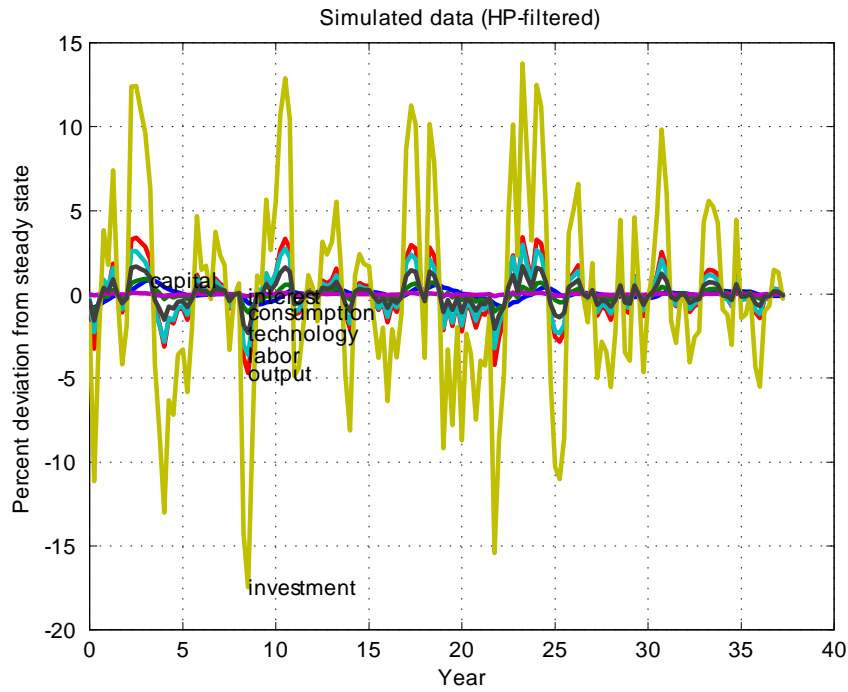Table 11.1: Business Cycles: Data and Model

Figure 11.3: Simulated Data from the Hansen Model

of the model for 40 periods, whose summary statistics (for a simulation of longer length) are displayed in table 11.1. The most outstanding fact from the simulation is the high volatility of investment. This is not a surprise, but shows up already in the impulse response function 11.2. In addition, the high volatility of investment is a salient fact of business cycles in the data: investment demand is by far the most volatile component of GDP.

11.3   Counterfactual Experiments

Set $\rho_z = 0$

   Make σ_z smaller

   Change utility function for leisure

Part IV

Welfare and Policy Questions

[Next version of these notes]

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Chapter 12

The Cost of Business Cycles

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