Exercises to Seminar 1
ECON 4310

A perfect foresight infinite horizon problem

In this question we will look at a discrete-time version of Ramsey's growth model.

1. The representative agent has utility

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

c_t is period t consumption and $\beta \in (0, 1)$ is the subjective discount factor. The period utility function is:

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$$

with $\theta > 1$. Every period the agent earns a wage w_t (its labor supply is exogenously set to 1 unit) and interests $r_t b_t$ from its bond holdings. It must pay τ_t in taxes. It will choose the stream $\{c_t, b_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + b_{t+1} = w_t + (1 + r_t)b_t - \tau_t \tag{2}$$

as well as an initial value for b_0. The agent also takes all future taxes, wage rates and interest rates as given.

   (a) Assume that the no-Ponzi condition holds (we will return to that later). Formulate the Lagrange problem for the agent with λ_t as the Lagrange multiplier for the budget constraint. Derive the first-order conditions for c_t and b_{t+1}, and combine these to solve for the consumption Euler equation. Interpret the condition.

(b) Use the Euler equation to find the intertemporal elasticity of substitution (i.e., the elasticity of substitution between consumption in period t and in period $t + 1$)

2. The representative firm will demand capital k_t and labor n_t to produce output y_t. It acts as a price-taking profit maximizer and has a Cobb-Douglas production function

$$y_t = k_t^{\alpha} n_t^{1-\alpha} \tag{3}$$

Capital is rented for r_t while labor costs w_t.

(a) Find the first-order conditions for the firm's optimization problem.

3. The government can raise lump-sum taxes τ_t, spend g_t and issue debt d_{t+1}. It must pay an interest rate r_t on its debt. It has a period-by-period budget constraint

$$g_t - d_{t+1} = \tau_t - (1 + r_t)d_t \tag{4}$$

(a) Use the government's budget constraint to find an expression for d_0. Subsitute for d_t iteratively ($t = 1, 2, 3, ..., T$) such that you find

$$d_0 = \sum_{t=0}^{T} \frac{\tau_t - g_t}{\Pi_{s=0}^{t}(1 + r_s)} + \frac{d_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} \tag{5}$$

(b) Interpret the assumption

$$\lim_{T\to\infty} \frac{d_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0$$

(c) Impose this, and also assume $d_0 = 0$. What is now the meaning of condition (5)?

(d) Do the same exercise for the representative agent's budget constraint and find the no-Ponzi condition for the household.

4. Assume that (5) holds for a given stream $\{\tau_t, g_t\}_{t=0}^{\infty}$. Then imagine that g_t is raised by Δ. This can either be financed by raising τ_t or d_{t+1}.

(a) Which method for financing the expenditure should the representative agent prefer? Answer by using the intertemporal budget constraints.

5. This model has three markets: The market for labor, the market for consumption, and the market for capital. By Walras' law, we know that the market clearing in two markets imply market clearing in the third (provided that all agents obey their budget constraints). Market clearing for labor requires

$$n_t = 1$$

while market clearing for capital requires

$$b_t - d_t = k_t$$

   (a) Characterize the competitive equilibrium using the first-order conditions, budget constraints and market clearing conditions.

6. Using the first welfare theorem, we know that the solution to the social planner's problem is equivalent to the competitive market equilibrium.

   (a) Write down the social planner's problem as a Lagrange problem.

   (b) Re-write the problem as a dynamic programming problem.

7. Dynamics: Using a phase diagram (i.e., a diagram with k_t on the x-axis and c_t on the y-axis) to figure out the optimal solution to the Ramsey problem, i.e., the optimal path of consumption for an initial value of k_t.

   (a) Using the resource constraint, draw a graph with all the combinations of (k_t, c_t) where the capital stock remains constant (i.e., $k_{t+1} = k_t$). For one particular value of the capital stock k_t, consider two values for c_t – one above the graph and one below it. For each choice of c_t, mark what direction capital will move (i.e., if $k_{t+1} > k_t$ or the other way around).

   (b) Consider now the Euler equation. Draw a graph with all the combinations of (k_t, c_t) such that consumption is constant (i.e., $c_{t+1} = c_t$). Pick some combinations of (k_t, c_t) and for each choice mark what direction consumption will move (i.e., if $c_{t+1} > c_t$ or the other way around).

   (c) Explain why the steady state by construction is at the intersection of the two graphs.

   (d) Using the phase diagram, illustrate what direction (k_t, c_t) will move (in all areas of the (k_t, c_t)-space).

(e) Draw the saddle path leading to the steady state. Explain what goes wrong if the individual chooses an initial consumption off the saddle path?

8. Consider the steady state where consumption is constant.

    (a) Characterize the steady state wage and real interest rate.

    (b) Compare the steady state real interest rate with the real interest rate that would prevail under the golden rule.