answers to seminar 4 (Sigurd)

1 The RBC model

In this problem set we will work through a simplified RBC model where we, as in the example covered in Lecture 11, will get an analytical solution. The general description of the model is that we have a social planner's problem described by:

$$\begin{aligned}
\max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}} E_t \sum_{s=t}^{\infty} \beta^{s-t}[\log c_s + \phi \log(1-n_s)] \\
s.t. \\
c_t + k_{t+1} &= A_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t \\
A_t &= Ae^{z_t} \\
z_t &= \rho z_{t-1} + \varepsilon_t \\
c_t &\geq 0 \\
k_{t+1} &\geq 0 \\
0 \leq n_t &\leq 1
\end{aligned}$$

with $k_0 > 0$ given. Can as before simplify by ignoring the conditions on n, c and k under 'normal' assumptions. We simplify the model by assuming:

$$\delta = 1$$

1. Derive the first-order conditions with resepect to c_t, n_t and k_{t+1}.

   - Method without Lagrangian: eliminate consumption by inserting the resource constraint in the utility function and differentiate to get first-order conditions wrt k_{t+1} and n_t
   - Method with Lagrangian: The Lagrangian in period t is

     $$L_t = E_t \left[\sum_{s=t}^{\infty} \beta^{s-t} \left(\log c_s + \phi \log(1-n_s)\right) - \lambda_s (c_s + k_{s+1} - A_s k_s^{\alpha} n_s^{1-\alpha}) \right]$$

     The first-order conditions are

     $$\begin{aligned}
     (1) c_t &: \quad \frac{1}{c_t} = \lambda_t \\
     (2) n_t &: \quad \frac{\phi}{(1-n_t)} = \lambda_t (1-\alpha) A_t \left(\frac{k_t}{n_t}\right)^{\alpha} \\
     (3) k_{t+1} &: \quad \lambda_t = E_t \left[\lambda_{t+1} \alpha A_{t+1} \left(\frac{k_{t+1}}{n_{t+1}}\right)^{\alpha-1} \right]
     \end{aligned}$$

Notice how the expectation operator E_t (the expectation conditional on information available in period t) drops out of foc (1) and (2). This is because there's no uncertainty about today's consumption, labor supply and saving (c_t, n_t, k_{t+1}) (they're based on the information available today). In contrast, future $c_{t+i}, n_{t+i}, k_{t+1+i}$ outcomes depend on the state of the world in the future, which is uncertain.

2. Combine the first-order conditions for c_t and k_{t+1} to find the *inter-temporal* (Euler) condition:

$$1 = \beta E_t \left(\alpha A_{t+1} \left(\frac{k_{t+1}}{n_{t+1}} \right)^{\alpha-1} \frac{c_t}{c_{t+1}} \right)$$

- **Solution:** Eliminate the Lagrange multiplier from foc (3) using foc (1). In a deterministic setting we would just update foc (1) one period ahead to get $\beta/c_{t+1} = \lambda_{t+1}$. However, in a stochastic setting future lagrange multipliers are uncertain. If we take the first order condition wrt c_{t+1}, we get the FOC

$$E_t \frac{\beta}{c_{t+1}} = E_t \lambda_{t+1}$$

The shadow value of wealth and marginal utility depends on the state of the world next period (productivity) which is unknown when we're in period t. It is okay to replace λ_{t+1} with β/c_{t+1} in foc (3), nevertheless. This will give you the inter-temporal condition.

- **bonus**: Since λ_{t+1} seems to be equal β/c_{t+1} only in expectation it is not trivial why the above solution works. *Short explanation*: Both β/c_{t+1} and λ_{t+1} depend on the state of the world next period. But conditional on the state, they will be equal. It follows that they're equal also in expectation. *long explanation*:For simplicity, suppose that in period t there are only two possible states of the world (low and high productivity) next period.[1] Let π_l denote the probability of a bad state and $\pi_h = 1 - \pi_l$ the probability of a good state. In a stochastic setting, goods are differentiated not only by time, but also by the state of the world. This means that consumption (and labor supply) in period $t+1$ when productivity is low (c^l_{t+1}) is a different good than consumption when productivity is high (c^h_{t+1}). We will also have a resource constraint and a corresponding lagrange multiplier for each state of the world, λ^l_{t+1} and λ^h_{t+1}. The key is to understand that in a stochastic problem we choose a sequence of allocations that gives optimal consumption, not only in each period, but for each possible state of the world in each period. Hence, in period t the planner decides on an optimal period $t+1$ consumption

[1] In our RBC model there's acually a continuum of possible states next period. But to make the argument simple, I consider only two.

plan, i.e. consumption in each possible state of the world in period t+1,[2] The FOCs are

$$\begin{aligned}
\pi^i \frac{\beta}{c^i_{t+1}} &= \pi^i \lambda^i_{t+1} \\
&\Leftrightarrow \\
\frac{\beta}{c^i_{t+1}} &= \lambda^i_{t+1} \text{ for } i = \{l, h\}
\end{aligned}$$

So conditional on state, the discounted marginal utility of period $t+1$ consumption is equal to the shadow value of period $t+1$ wealth. It follows that it also holds in expectation. Next write out the expectation in foc (3) in our simplified example to get

$$\begin{aligned}
\frac{1}{c_t} &= \left[\pi^l \lambda^l_{t+1} \alpha A^l_{t+1} \left(\frac{k_{t+1}}{n^l_{t+1}}\right)^{\alpha-1} + \pi^h \lambda^h_{t+1} \alpha A^h_{t+1} \left(\frac{k_{t+1}}{n^h_{t+1}}\right)^{\alpha-1}\right] \\
&= \left[\pi^l \frac{\beta}{c^l_{t+1}} \alpha A^l_{t+1} \left(\frac{k_{t+1}}{n^l_{t+1}}\right)^{\alpha-1} + \pi^h \frac{\beta}{c^h_{t+1}} \alpha A^h_{t+1} \left(\frac{k_{t+1}}{n^h_{t+1}}\right)^{\alpha-1}\right] \\
&= \beta E_t \left[\frac{1}{c_{t+1}} \alpha A_{t+1} \left(\frac{k_{t+1}}{n_{t+1}}\right)^{\alpha-1}\right]
\end{aligned}$$

If interested, check out the end of the document for an example in a two period economy

3. Show that you can combine the first-order conditions for c_t and n_t to find the *intratemporal* condition:

$$\phi \frac{c_t}{1-n_t} = (1-\alpha) A_t \left(\frac{k_t}{n_t}\right)^{\alpha}$$

- **Solution:** Eliminate the Lagrange multiplier in foc (2) using foc (1).

4. The intra- and intertemporal conditions and the resource constraint are defining the solution. Let us conjecture a solution of the form:

$$\begin{aligned}
n_t &= \bar{n} \\
c_t &= \gamma_c A_t k_t^{\alpha} \bar{n}^{1-\alpha} \\
k_{t+1} &= \gamma_k A_t k_t^{\alpha} \bar{n}^{1-\alpha}
\end{aligned}$$

   (a) Insert the solutions for c_t, c_{t+1} and n_t in the Euler equation. Show that it gives

$$1 = \beta E_t \left(\alpha \frac{A_t k_t^{\alpha} \bar{n}^{1-\alpha}}{k_{t+1}}\right)$$

[2] When we in period t differentiate wrt c_{t+1} we are doing a short cut. What we should do is to differentiate wrt next period consumption conditional on each possible state of the world next period

Then insert the solution for k_{t+1}. Verify that γ_k must satisfy

$$\gamma_k = \alpha\beta$$

- **Solution:** First we insert for c_t, c_{t+1} and n_t. In the Euler equation we get:

$$\begin{aligned} 1 &= \beta E_t \left(\alpha A_{t+1} \left(\frac{k_{t+1}}{\bar{n}} \right)^{\alpha-1} \frac{\gamma_c A_t k_t^{\alpha} \bar{n}^{1-\alpha}}{\gamma_c A_{t+1} k_{t+1}^{\alpha} \bar{n}^{1-\alpha}} \right) \\ &\Leftrightarrow \\ 1 &= \beta E_t \left(\alpha \frac{A_t k_t^{\alpha} \bar{n}^{1-\alpha}}{k_{t+1}} \right) \end{aligned}$$

Then insert for k_{t+1}. Follows that

$$\begin{aligned} 1 &= \alpha\beta E_t \left(\frac{A_t k_t^{\alpha} \bar{n}^{1-\alpha}}{\gamma_k A_t k_t^{\alpha} \bar{n}^{1-\alpha}} \right) \\ &\Leftrightarrow \\ 1 &= \alpha\beta \frac{1}{\gamma_k} \\ \gamma_k &= \alpha\beta \end{aligned}$$

(b) Next, use the resource constraint to find

$$\gamma_c = 1 - \gamma_k$$

- **Solution:** Resource constraint

$$\begin{aligned} c_t &= y_t - k_{t+1} \\ &\Leftrightarrow \\ \gamma_c y_t &= y_t - \gamma_k y_t = (1 - \gamma_k) y_t \\ &\Leftrightarrow \\ \gamma_c &= 1 - \gamma_k \end{aligned}$$

(c) Finally use the intratemporal condition to confirm that the constant value of labor supply is

$$\bar{n} = \frac{1 - \alpha}{1 - \alpha + \phi(1 - \gamma_k)}$$

- **Solution:** When we insert for c_t in the intratemporal condition we get

$$\phi \frac{\gamma_c A_t k_t^{\alpha} \bar{n}^{1-\alpha}}{1 - \bar{n}} = (1 - \alpha) A_t \left(\frac{k_t}{\bar{n}} \right)^{\alpha}$$

After simplifying we have

$$\phi \frac{\gamma_c \bar{n}}{1 - \bar{n}} = (1 - \alpha)$$

Then solve for $\bar{n}$.

5. We want to look at the impulse-response function for a shock to productivity. To find the impulse-response function we:

   - Start out in steady state
   - Shock productivity in period t (i.e. $\varepsilon_t = \Delta$) and keep the shocks equal to zero in future periods ($\varepsilon_{t+i} = 0$ for $i = 1, 2, ...$).
   - Then plot the response of k_{t+1+i}, c_{t+i} and n_{t+i} for $i = 0, 1, 2, 3, ..., k$.

   Assume $A = 1$, $\phi = 1$, $\beta = 0.99$, $\alpha = 0.33$, $\rho = 0.95$ and use $\Delta = 1$. Using Excel or some other software, draw the impulse-response functions. Start out by solving for steady state. The steady state in a model with shocks is found by setting the shocks to zero $\Rightarrow z_t = 0$ (as long as $\rho < 1$) and $A_t = A$. Find the steady state capital stock from the capital equation

$$\begin{aligned} k_{t+1} &= \gamma_k A k_t^{\alpha} \bar{n}^{1-\alpha} \\ k_{ss} &= \gamma_k A k_{ss}^{\alpha} \bar{n}^{1-\alpha} \\ k_{ss} &= (\gamma_k A)_{ss}^{\frac{1}{1-\alpha}\alpha} \bar{n} \end{aligned}$$

   Then we can compute $y_{ss} = A k_{ss}^{\alpha} \bar{n}^{1-\alpha}$, and $c_{ss} = \gamma_c y_{ss}$. The next step is to calculate the evolution of productivity (note that $z_{t-1} = 0$ since we assume we are in SS when we enter period t)

$$\begin{aligned} z_t &= \rho z_{t-1} + \varepsilon_t = 1 \\ z_{t+i} &= \rho z_{t+i-1} \text{ for } i = 1, 2, ... \\ A_{t+i} &= A e^{z_{t+i}} \text{ for } i = 0, 1, 2... \end{aligned}$$

   Finally, compute the capital stock and consumption

$$\begin{aligned} k_t &= k_{ss} \\ k_{t+i} &= \gamma_k A_t k_{t+i-1}^{\alpha} \bar{n}^{1-\alpha} \text{ for } i = 1, 2, .. \\ c_{t+i} &= \gamma_c A_t k_{t+i-1}^{\alpha} \bar{n}^{1-\alpha} \text{ for } i = 0, 1, 2, .. \end{aligned}$$

6. For a variable x we let $\hat{x}_t$ denote its percentage deviation from steady state, i.e. $\hat{x}_t = (x_t - x^*)/x^*$. Find $\hat{c}_t$ and $\hat{k}_{t+1}$. (*Hint: You should get* $\hat{c}_t = \hat{k}_{t+1}$.

   - **Solution:** We have

$$\begin{aligned} \widehat{c_t} &= \frac{c_t}{c_{ss}} - 1 = \frac{\gamma_c y_t}{\gamma_c y_{ss}} - 1 = \widehat{y_t} \\ \widehat{k}_{t+1} &= \frac{k_{t+1}}{k_{ss}} - 1 = \frac{\gamma_k y_t}{\gamma_k y_{ss}} - 1 = \widehat{y_t} \end{aligned}$$

7. Compare the percentage increases in investment and consumption that we get for this model, with the pattern we saw in Lecture 13. Find one important reason for why the pattern is so different.

- **Solution:** Investment, consumption, output and capital have the same volatility (percentage deviation from SS). In the pattern from lecture 13, investment is more volatile than output and consumption less. The reason we get equal volatility in our model is because of log utility and full depreciation, which gives constant labor supply and savings rate (just as in the Solow model).

2 Labor supply

Let us look more carefully at the labor supply decision. For this problem we consider a more general utility function:

$$u(c, 1-n) = \log c + \phi \frac{(1-n)^{1-\theta} - 1}{1-\theta}$$

1. Try to show that

$$\lim_{\theta \to 1} \frac{(1-n)^{1-\theta} - 1}{1-\theta} = \log(1-n)$$

Hint: Use L'Hopital's rule.

- **Solution:** The limit is "0/0", so we must use L'Hopital's rule. Differentiate the nominator and denominator separately:

$$\lim_{\theta \to 1} \frac{(1-n)^{1-\theta} - 1}{1-\theta} = \lim_{\theta \to 1} \frac{-(1-n)^{1-\theta}\log(1-n)}{-1} = \log(1-n)$$

Note: Same reason the special case of CRRA utility with CRRA=1 gives log utility.

2. Update the intratemporal condition to the case with the more general utility function. Let us define the Frisch elasticity of leisure as the elasticity of $1-n$ with respect to the wage rate, holding the marginal utility of consumption constant. Use the intratemporal condition to verify that the Frisch elasticity of leisure is given by $-1/\theta$.

- **Solution:** Let El_w denote the elasticity wrt w. The new intratemporal condition is:

$$\begin{aligned} \phi(1-n)^{-\theta} c &= w \\ (1-n) &= w^{-\frac{1}{\theta}} (c\phi)^{\frac{1}{\theta}} \end{aligned}$$

Holding marginal utility of consumption constant we treat $(c\phi)^{\frac{1}{\theta}}$ as a constant. It follows that $El_w(1-n) = -\frac{1}{\theta}$

3. Then try to find the Frisch elasticity of *labor supply* using the same condition. Verify that it is not constant.

- **Solution:** From the rules of elasticity[3] we get that $El_w(1-n) = -\frac{n}{1-n}El_w(n)$ so we get

$$\begin{aligned} -\frac{1}{\theta} &= -\frac{n}{1-n}El_w(n) \\ &\Leftrightarrow \\ El_w(n) &= \frac{1-n}{n}\frac{1}{\theta} \end{aligned}$$

4. Use your answers to the last two questions to say something about the difference between utility functions defined in terms of leisure (such as the one above) and utility functions defined in terms of labor supply, such as

$$u(c,n) = \log c - \phi\frac{n^{1+\theta}}{1+\theta}$$

(This question should help you understand why the Frisch elasticity is sometimes defined in terms of leisure and other times in terms of labor supply.)

   - **Solution:** The first has a constant Frisch elasticiy of leisure. The other has a constant Frisch elasticity of labor..

5. Finally, let us look at a problem where labor supply is a choice along both the extensive and intensive margin. We are looking at an economy consisting of *two* individuals: Nick and Adam. They are along most dimensions identical: Both have utility over consumption and labor supply given by:

$$u(c_i, n_i) = \frac{c_i^{1-\sigma}}{1-\sigma} - \phi\frac{n_i^{1+\theta}}{1+\theta}$$

where $i = N, A$ indicates Nick or Adam. Note that the parameters in the utility function are the same. It is a one-period model, and both are maximizing utility subject to a budget constraint:

$$c_i = wn_i + b_i$$

(notice that they face the same wage rate). b_i is the exogenous assets that they have available. The only *difference* between them is that Nick has a flexible job where he can choose his number of hours freely, subject to the constraint that $0 \le n_N \le 1$. Adam, on the other hand, can only choose to work full-time or not work at all, i.e. $n_A \in \{0,1\}$. Make sure you understand that Nick's Frisch elasticity of labor supply is $1/\theta$.

   (a) For Nick, we find labor supply in the usual way (assuming an interior solution). Use the budget constraint to insert for c_i in the utility function. Find the first-order condition (the intratemporal optimality condition).

[3] $El_x(f(x) - g(x)) = \frac{fEl_x(f) - gEl_x(g)}{f-g}$ and $El_x(1) = 0$.

- **Solution:**

$$\begin{aligned} c_N^{-\sigma} w - \phi n_N^{\theta} &= 0 \\ n_N &= w^{\frac{1}{\theta}} \left(\frac{c_N^{-\sigma}}{\phi} \right)^{\frac{1}{\theta}} \end{aligned}$$

(b) Then find the decision-rule for Adam's labor supply (define the wage rate w_A^* that makes Adam indifferent between working and not working).

- **Solution:**

$$\frac{(w_A^* + b_A)^{1-\sigma}}{1-\sigma} - \phi \frac{1}{1+\theta} = \frac{(b_A)^{1-\sigma}}{1-\sigma}$$

Possible for solve for w_A^*, but most important to understand $u(w_A^* + b_A, 1) = u(b_A, 0)$. So the labor supply for Adam is

$$n_A = \begin{cases} 0 \text{ if } w < w_A^* \\ 1 \text{ if } w \geq w_A^* \end{cases}$$

(c) Calculate the (macro) Frisch elasticity of labor supply for $w < w_A^*$.

- **Solution:** For $w < w_A^*$, total labor supply is simply what Nick supplies. Hence the percentage change in macro labor supply is just the change in his. Macro = micro Frisch elasticity.

(d) Calculate the (macro) Frisch elasticity of labor supply for $w > w_A^*$.

- **Solution:** For $w > w_A^*$, total labor supply is $1 + n_N$. Macro frisch elasticity is therefore

$$\frac{n_N}{1+n_N} El_w(n_N) = \frac{n_N}{1+n_N} \frac{1}{\theta} < \frac{1}{\theta}$$

(e) Explain (but do not calculate) what the Frisch elasticity is at the point when w is just below w_A^*.

- **Solution:** At the point when n_A goes from 0 to 1, the Frisch elasticity is 'infinite': There is no derivative since we jump from one part to another.

(f) What is the lesson for the relation between micro and macro Frisch elasticities?

- The macro elasticity is not the same as the micro elasticity. In general the macro elasticity is the weighted average of the micro elasticities. When only Nick works, macro elasticity equals Nick's elasticity. When both Nick and Adam works, the macro elasticity equals the weighted average of Nick's and Adam's elasticities, with weights equal to N and A's fraction of total labor supply. But when A works, he works $n_A = 1$, so his elasticity is zero. Hence, macro elasticity equals Nick's elasticity weighted by Nick's fraction of total labor supply.

- The macro elasticity can be larger than the micro elasticity when wage changes makes individuals move in and out of work (movements along the extensive margin). Micro estimates tend to find a relatively low elasticity. These elasticities are typically estimated on individuals observed working before and after a wage change, hence they don't capture the extensive margin. One lesson is that we can have both a low micro elasticity (changes along the intensive margin) and a large macro elasticity (due to changes along the extensive margin)

3 Extra: two period example of the stochastic Euler equation

Suppose we have the following two period model with instantaneous utility function $u(c_t)$. In period 1 the resource constraint is simply

$$c_1 + k_2 = A_1 k_1^{\alpha}$$

and in period 2 the resource constraint is

$$c_2 = A_2 k_2^{\alpha}$$

When starting in period 1 the next period productivity level A_2 is stochastic. Suppose it can take two possible values, low (A_2^l) with probability π_l and high (A_2^h) with probability ($\pi_h = 1 - \pi_l$). The period 1 maximization problem involves choosing period 1 consumption c_1, next period capital k_2, and state-dependent period 2 consumption, i.e., optimal consumption in both possible states of the world, c_2^l and c_2^h. Let λ_1 denote the Lagrange multiplier on the period 1 resource constraint, and $\left\{\pi_l \lambda_2^l, \pi_h \lambda_2^h\right\}$ the multipliers for the period 2 resource constraint,[4] one for each possible realization of A_2. In period 1 we don't know the period 2 productivity and consumption outcomes, but if the state is bad consumption is given by c_2^l and correspondingly c_2^h if the state is good. The Lagrangian is then

$$\begin{aligned} L(c_1, k_2, c_2^l, c_2^h) &= u(c_1) + \beta(\pi_l u(c_2^l) + \pi_h u(c_2^h)) - \lambda_1(c_1 + k_2 - A_1 k_1^2) - \pi_l \lambda_2^l (c_2^l - A_2^l k_2^{\alpha}) - \pi_h \lambda_2^h (c_2^h - A_2^h k_2^{\alpha}) \\ &= E_1\left[\sum_{j=1}^{2} \beta^{j-1} u(c_j)\right] - E_1\left[\sum_{j=1}^{2} \lambda_j (c_j - A_j k_j^{\alpha})\right] \end{aligned}$$

where the first term in the last expression is expected utility, i.e. the sum over all possible utility outcomes $u(c_2^i)$, weighted by their relevant probabilities π^i (Note that there's only one possible outcome for period 1 utility $u(c_1)$). The

[4] The presence of π_l and π_h in the multiplier is an innocent normalization that makes the lagrange expression simpler to work with. Note that we can drop this, and still get the same optimality conditions.

second term is the resource constraint weighted by the multipliers. Since we chose to normalize the multipliers using the probabilities for period 2 states, we can write this expression as an expectation too, giving the Lagrangian in compact form:

$$L = E_1 \left[\sum_{j=1}^{2} \left(\beta^{j-1} u(c_j) - \lambda_j (c_j - A_j k_j^{\alpha}) \right) \right]$$

which is the two period version of the infinite horizon Lagrangian. The first order conditions are

$$\begin{aligned}
(1)\ c_1 &: \quad u'(c_1) = \lambda_1 \\
(2)\ c_2^l &: \quad \beta u'(c_2^l) = \lambda_2^l \\
(3)\ c_2^h &: \quad \beta u'(c_2^h) = \lambda_2^h \\
(4)\ k_2 &: \quad \lambda_1 = \pi \lambda_2^h \alpha A_2^l k_2^{a-1} + \pi \lambda_2^h \alpha A_2^h k_2^{\alpha-1}
\end{aligned}$$

Finally, insert condition (1)-(3) in (4) to get

$$\lambda_1 = \beta \pi_l u'(c_2^l) \alpha A_2^l k_2^{a-1} + \beta \pi_h u'(c_2^h) \alpha A_2^h k_2^{\alpha-1}$$

or more compact as

$$u'(c_1) = \beta E_1 \left[u'(c_2) \alpha A_2 k_2^{\alpha-1} \right]$$