Seminar 5 answers (Sigurd)

1 Saving and risk

Consider an economy with constant population, $N_t = N$. The utility function of an agent who is young in period t is

$$u_t = \ln(c_{y,t}) + E_t \ln(c_{o,t+1})$$

where $c_{y,t}$ and $c_{o,t+1}$ are his consumption when young and old. There is no real capital in the economy. One unit of labor produces one unit of goods. Young agents are always endowed with $\omega_{y,t} = 1$ units of labor. The endowments of old agents, $\omega_{o,t+1}$ are uncertain. Half of them get $\omega_{o,t+1} = 1 + \Delta$, the other half $\omega_{o,t+1} = 1 - \Delta$. Individuals learn which group they belong to at the start of the period when they are old. They have no indication before that. Thus, ex ante, all agents are identical. There is only one asset, private lending with an interest rate r_{t+1} from period t to $t + 1$. No insurance against endowment risk is sold. Note that there is no aggregate risk.

1. Write down the period by period budget constraints for an agent that is young in period t. Let the assets that a representative young consumer carries over from period t to period $t + 1$ be denoted a_{t+1}.

   - **Solution:** Budget constraints are

   $$\begin{aligned} c_{y,t} + a_{t+1} &= \omega_{y,t} \\ c_{o,t+1} &= \omega_{o,t+1} + (1 + r_{t+1})a_{t+1} \end{aligned}$$

2. Derive the first order conditions for maximum utility.

   - **Solution:** Use the budget constraints to insert for consumption in both periods:

   $$u_t = \ln(1 - a_{t+1}) + E_t \ln(\omega_{o,t+1} + (1 + r_{t+1})a_{t+1})$$

   so the maximization problem is

   $$\max_{a_{t+1}} \{\ln(1 - a_{t+1}) + E_t \ln(\omega_{o,t+1} + (1 + r_{t+1})a_{t+1})\}$$

   First-order condition wrt a_{t+1} is:

   $$\begin{aligned} -\frac{1}{1 - a_{t+1}} + E_t \left(\frac{1 + r_{t+1}}{\omega_{o,t+1} + (1 + r_{t+1})a_{t+1}} \right) &= 0 \\ &\Leftrightarrow \\ \frac{1}{c_{y,t}} &= E_t \left(\frac{1}{c_{o,t+1}} (1 + r_{t+1}) \right) \\ &\Leftrightarrow \\ \frac{1}{c_{y,t}} &= (1 + r_{t+1}) E_t \left(\frac{1}{c_{o,t+1}} \right) \end{aligned}$$

where the last expression follows since r_{t+1} is not stochastic (can pull it outside the expectation).

3. What is the equilibrium level of a_{t+1}? (No algebra needed).

   - **Solution:** Capital market clearing requires aggregate assets=aggregate capital stock. This economy has no capital$\Rightarrow$ total assets must be zero. In such an economy, the only way for a period t young agent to save (or borrow) is to find an agent (young or old) that is willing to borrow (or lend). But young agents are identical $\Rightarrow$everyone choose the same next period asset (i.e. either all young agents are lenders, or they are all borrowers). And period t old agents cannot borrow, since they are not alive next period, and they don't want to save (not optimal). Capital market clearing thus implies that individual asset holding is zero, $a_{t+1} = 0$.

4. Use the answers to the two preceding questions to determine the interest rate r_{t+1}

   - **Solution:** The equilibrium interest rate is the r_{t+1} that makes it optimal to choose $a_{t+1} = 0$. The FOC gives us the optimal a_{t+1} as a function of the interest rate. To find this interest rate, set $a_{t+1} = 0$ in the FOC and solve for r_{t+1}:

     $$1 + r_{t+1}E_t\left(\frac{1}{\omega_{o,t+1}}\right) = 1$$

     Then we use what we know about the distribution of $\omega_{o,t+1}$:

     $$\frac{1+r_{t+1}}{2}\left(\frac{1}{1+\Delta}\right) + \frac{1+r_{t+1}}{2}\left(\frac{1}{1-\Delta}\right) = 1$$

     Can write this as

     $$(1+r_{t+1})\left(\frac{1}{1+\Delta} + \frac{1}{1-\Delta}\right) = 2$$

     or

     $$(1+r_{t+1}) = (1+\Delta)(1-\Delta)$$

     Finally, we get

     $$r_{t+1} = -\Delta^2 < 0$$

5. How does the interest rate and the equilibrium amount of saving depend on the degree of income risk? How do you explain what goes on here in relation to common theories about saving?

- **Solution:** what is Δ^2. Endowment when old is $w_{o,t+1} = 1 + \varepsilon_{t+1}$, where $E(\varepsilon_{t+1}) = \frac{1}{2}(-\Delta) + \frac{1}{2}\Delta = 0$, and $var(\varepsilon_{t+1}) = E(\varepsilon_{t+1}^2) - [E(\varepsilon_{t+1})]^2 = E(\varepsilon_{t+1}^2) = \frac{1}{2}(-\Delta)^2 + \frac{1}{2}\Delta^2 = \Delta^2$. So Δ^2 is the variance (risk) of the endowment when old. The equilibrium amount of saving is always zero, but the interest rate is negatively related to risk. The more risk, the lower interest rate. Why? Common theory about saving: save to smooth consumption (OLG: save for old age). Risk provides an additional motive to save for this reason. Suppose we shut down risk ($\Delta = 0$). The interest rate would be zero. Since $\beta = 1$ and $r_{t+1} = 0$ the young would like perfect consumption smoothing $c_{y,t} = c_{o,t+1}$. Since income while young=income while old, there's no incentive to save. If we now add risk $\Delta > 0$, the young would like to save for *precautionary* reasons. Precautionary saving is a way to self-insure against adverse income shocks when old, i.e. you build a buffer stock of financial assets, a buffer you can use to smooth consumption in case $\omega_{o,t+1}$ is low. If risk increases, the precautionary motive is stronger, and we get excess supply of savings if the interest rate remains constant. To restore equilibrium we therefore need to reduce the supply of saving, which is achieved by reducing the interest rate.

6. Would you get the same result if the utility function were quadratic?

   - **Solution:** No. Why not? Since then marginal utility is linear in consumption. Quadratic instantaneous utility $u = c - \frac{b}{2}c^2$gives

$$\begin{aligned} u_t &= u(c_{y,t}) + E_t u(c_{o,t+1}) \\ &= c_{y,t} - \frac{b}{2}c_{y,t}^2 + E_t\left[c_{o,t+1} - \frac{b}{2}c_{o,t+1}^2\right] \end{aligned}$$

The FOC for a_{t+1} is

$$\begin{aligned} (1 - bc_{y,t}) &= (1 + r_{t+1})E_t\left[1 - bc_{o,t+1}\right] \\ &\Leftrightarrow \\ (1 - b(1 - a_{t+1})) &= (1 + r_{t+1})E_t\left[1 - b(\omega_{o,t+1} + (1 + r_{t+1})a_{t+1})\right] \\ &\Leftrightarrow \\ (1 - b(1 - a_{t+1})) &= (1 + r_{t+1})(1 - b(1 + r_{t+1})a_{t+1})) - (1 + r_{t+1})bE_t\left[\omega_{o,t+1}\right] \end{aligned}$$

Hence, only expectation matters $E_t\left[\omega_{o,t+1}\right]$, and the degree of risk does not influence the saving decision. For $a_{t+1} = 0$ and $E_t\left[\omega_{o,t+1}\right] = 1$ this gives

$$\begin{aligned} 1 &= (1 + r_{t+1}) \\ r_{t+1} &= 0 \end{aligned}$$

With linear marginal utility, certainty equivalence holds, i.e. the agent behaves as if the stochastic income is equal to expected income with certainty. Therefore, with quadratic utility, endowment risk does not matter. Why? Suppose we start out with the an optimal saving decision, satisfying the Euler equation.

$$u'(c_{y,t}) = (1 + r_{t+1})E_t u'(c_{o,t+1})$$

Now, suppose uncertainty about endowments when old increases, but the mean is kept constant. The expected marginal utility $E_t u'(c_{o,t+1})$ is unchanged if u' is linear (which is true if $u''' = 0$). In contrast, if marginal utility is convex ($u''' > 0$) expected marginal utility increases. Hence, the right hand side of the Euler equation increases and the FOC no longer holds. The agent therefore reduces consumption when young (increased saving), which make $u'(c_{y,t})$ increase and $E_t u'(c_{o,t+1})$ decrease

- Risk aversion is determined by the concavity of the utility function (u'') and is thus not the same as precautionary saving which is determined by the convexity of marginal utility (u''').
- Note that in a more general model (more than 2 period life-cycle), if we introduce credit constraints (constraints on how much you can borrow from one period to another), increased saving in response to increased uncertainty is possible, even with quadratic utility. Intuition: higher risk increases the probability of being credit constrained in the future. In response, you might want to save more to reduce the probability of becoming constrained. In general, both credit constraints and convex marginal utility of consumption can explain buffer stock saving.[1] cf. the chapter on consumption in Romer's book for more on this.

2 Risky assets

We are looking at a consumer who lives for two periods. He starts life with no assets and leaves no bequests. His consumption in the two periods is C_0 and C_1. His earned income is Y_0 in the first period and Y_1 in the second period. In the first period he has the opportunity to invest in two assets which yield gross returns in the second period $1 + r_a$ and $1 + r_b$. The amount he invests in the first asset is A_a, in the second A_b. Y_1, r_a and r_b are stochastic.

1. Write down the budget equations for the consumer for the two periods.

   - **Solution:**

$$C_0 + A_a + A_b = Y_0$$
$$C_1 = Y_1 + (1 + r_a)A_a + (1 + r_b)A_b$$

[1] actually, in our two period OLG model, we could get this even with quadratic utility. Suppose risk suddenly increased so much that, if the agent did not save more for old age, he would have zero (or negative) consumption if the bad income shock is realized. Clearly, the reaction when young is to increase saving to make sure this doesn't happen.

2. The consumer's utility function is

$$U = u(C_0) + \frac{1}{1+\rho} E_0 u(C_1)$$

where ρ is the discount rate and

$$u(C) = \frac{C^{1-\theta}}{1-\theta}$$

Derive the first-order conditions for maximum utility and give a verbal interpretation of them.

- **Solution:** Insert for budget constraints in the utility function. Gives

$$U = u(Y_0 - A_a - A_b) + \frac{1}{1+\rho} E_0 u(Y_1 + (1+r_a)A_a + (1+r_b)A_b)$$

First-order conditions with respect to A_a and A_b:

$$u'(C_0) = \frac{1}{1+\rho} E_0 \left[(1+r_a)u'(C_1)\right]$$
$$u'(C_0) = \frac{1}{1+\rho} E_0 \left[(1+r_b)u'(C_1)\right]$$

Both conditions are familiar Euler equations. The left hand side is the utility loss from investing an additional unit of resources. The right hand sides are the discounted expected utility gain next period from additional unit of investment, in either assets. Hence, on the margin the consumer is indifferent between investing or not, and whether to invest in asset a or b.

3. Assume (for this question only) that the two assets have the same expected return. Show that the first-order conditions in this case imply that

$$cov(u'(C_1), 1+r_a) = cov(u'(C_1), 1+r_b)$$

Which asset will be most in demand, a or b?

- **Solution:** Combine the two first-order conditions to get:

$$E_0 \left[(1+r_a)u'(C_1)\right] = E_0 \left[(1+r_b)u'(C_1)\right]$$

Then use the fact that $E(XY) = EXEY + cov(X,Y)$,

$$E(1+r_a)E(u'(C_1)) + cov(u'(C_1), 1+r_a) = E(1+r_b)E(u'(C_1)) + cov(u'(C_1), 1+r_b)$$

Since (for this question) $E(1+r_a) = E(1+r_b)$ this simplifies to

$$cov(u'(C_1), 1+r_a) = cov(u'(C_1), 1+r_b)$$

- Three observations: (1) For a given variance on asset returns, the larger(smaller) fraction of savings invested in asset a the more (less) consumption positively covaries with the return on that asset. (2) For a given fraction, the higher variance on asset a the more C_1 covaries positively with the return on asset a. (3) Asset returns vary positively with consumption, hence it varies negatively with marginal utility (when return is high, consumption is high and marginal utility is low)
- Suppose that r_a and r_b are independent of Y_1. Suppose $var(1+r_a) > var(1+r_b)$. If you invest the same amount in both assets,

$$cov(u'(C_1), 1 + r_a) < cov(u'(C_1), 1 + r_b)$$

  Hence, you have to increase investment in asset b to get equality. In general you'd like to invest in both assets to diversify risk (insure against risk). if risk differs between assets, you invest less in the more risky asset.
- Suppose that $var(1 + r_a) = var(1 + r_b)$ but that the return on asset a is positively correlated with Y_1. If you invest the same amount in both assets

$$cov(u'(C_1), 1 + r_a) < cov(u'(C_1), 1 + r_b)$$

  since when the return on asset a is high (low), income also tends to be high (low), and thus marginal utility of consumption low (high). You have to increase investment in asset b to get equality. In general, to insure against risk you should invest less in an asset that is positively correlated with labor income and more in an asset that is negatively correlated with labor income.

4. Suppose the consumer earns his income from farming and that one of the assets is shares in a food-processing firm that buys its raw materials from his and similar farms. Should he buy shares in this firm and how much?

   - **Solution:** The most natural is probably to assume that the farmer's income is *negatively* correlated with the shares in the food processing firm. Why? The price of raw materials is determined on the world market, and tends to be volatile. This gives fluctuation in the farmer's income. On the other hand, the food processing firm will go well when the price of input is low and the value of the shares goes up. It is not possible to give an exact answer unless we know his preferences and the risk properties of this and other assets he can invest in. But in general he should buy shares in this firm as an *insurance against fluctuations in the price of raw materials*. How much? until the FOC is satisfied. If you try to argue that the share price is *positively* correlated with the farmer's income (If the food-processing industry goes well, the demand for the farmer's products go up), then he should tend to avoid these shares.

5. Assume now that asset a is risk-free. How does this change the first-order conditions?

   - **Solution:** In this case $1 + r_a$ is no longer stochastic. Combining the two first order conditions gives

$$
\begin{aligned}
(1 + r_a)E_0\left[u'(C_1)\right] &= E_0\left[(1 + r_b)u'(C_1)\right] \\
&\Leftrightarrow \\
(1 + r_a)E_0\left[u'(C_1)\right] &= E(1 + r_b)E(u'(C_1)) + cov(u'(C_1), 1 + r_b) \\
&\Leftrightarrow \\
E(r_b) - r_a &= -\frac{cov(u'(C_1), 1 + r_b)}{E(u'(C_1))}
\end{aligned}
$$

6. Now, assume that the economy is inhabited by a large number of consumers identical to the one we have studied. Explain how the first-order conditions from question 5 can then be used to determine the expected excess return on the risky asset.

   - **Solution:** if return on asset b is positively correlated with marginal utility of consumption, it provides insurance. Thus the excess return is negative. The interpretation is that we pay an insurance premium (in terms of lower return). If asset b is negatively correlated with marginal utility, consumers expose themselves to risk by buying asset b, which must be compensated for by higher expected return. Otherwise, nobody would buy asset b.