This note covers the parts of problem set 1 and 2 we did not solve at the seminars. In addition, there is a short discussion of the OLG model from ps2 at the end (Sigurd)

Problem set 1, exercise 6b

Given an initial capital stock $k_0 > 0$ the planner maximizes the representative agent's utility subject to the economy's resource constraint and non-negativity constraint on consumption and capital. The sequence of government consumption $\{g_t\}_{t=0}^{\infty}$ is exogenous.

$$\begin{aligned} & \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & s.t. \\ k_{t+1} &= k_t^{\alpha} + k_t - c_t - g_t \quad t = 0, 1, 2... \\ k_t &\geq 0, \quad c_t \geq 0 \end{aligned}$$

The shape of the instantaneous utility function $u(c)$ and the production function, implies that the non-negativity constraints will not bind, so let's ignore them.[1] Replacing c_t in the objective function using the resource constraint we define the indirect utility function (value function)

$$V_0(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(k_t^{\alpha} + k_t - g_t - k_{t+1})$$

which gives the maximal utility in period 0 given the initial capital stock k_0. Accordingly, $V_j(k_j)$ gives the maximal utility in period j given the capital stock in period j (think of this as an optimization problem starting in period j rather than period 0). Now, re-write $V_0(k_0)$ as

$$\begin{aligned} V_0(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(k_t^{\alpha} + k_t - g_t - k_{t+1}) \right\} \\ &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ u(k_0^{\alpha} + k_0 - g_0 - k_1) + \sum_{t=1}^{\infty} \beta^t u(k_t^{\alpha} + k_t - g_t - k_{t+1}) \right\} \\ &= \max_{k_1} \left\{ u(k_0^{\alpha} + k_0 - g_0 - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \left\{ \sum_{t=1}^{\infty} \beta^{t-1} u(k_t^{\alpha} + k_t - g_t - k_{t+1}) \right\} \right\} \\ &= \max_{k_1} \{ u(k_0^{\alpha} + k_0 - g_0 - k_1) + \beta V_1(k_1) \} \end{aligned}$$

This holds for any period t, and we get the Bellman equation

$$V_t(k_t) = \max_{k_{t+1}} \{ u(k_t^{\alpha} + k_t - g_t - k_{t+1}) + \beta V_{t+1}(k_{t+1}) \}$$

[1] Implicitly, we also assume that the exogenous government consumption is not too large. so that, given the initial capital stock, strictly positive consumption in all periods is feasible.

The idea is the following: In period t, choose the next period capital stock k_{t+1}, conditional on optimal behavior in the future (k_{t+1} will be the initial capital for the maximization problem starting in period $t+1$ and the value of leaving k_{t+1} for the future is then captured by the value function $V_{t+1}(k_{t+1})$). For simplicity, assume constant government consumption $g_t = g$ in all periods. The problem is then time-invariant, which implies that $V_t(k) = V_{t+1}(k)$. Why? If we start out in period $t = 0$ with a capital stock of $k_0 = k^*$, we get maximal utility $V_0(k_0)$. Now, suppose we start out in period s with the same initial capital stock $k_s = k^*$. Since it is an infinite horizon problem, the indirect utility will be the same in both cases,[2] i.e.

$$V_0(k^*) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(k_t^\alpha + k_t - g - k_{t+1}) \right\} = \max_{\{k_{t+1}\}_{t=s}^{\infty}} \left\{ \sum_{t=s}^{\infty} \beta^{t-s} u(k_t^\alpha + k_t - g - k_{t+1}) \right\} = V_s(k^*)$$

and we can drop the time subscript on the value functions

$$V(k_t) = \max_{k_{t+1}} \{u(k_t^\alpha + k_t - g_t - k_{t+1}) + \beta V(k_{t+1})\}$$

Problem set 1, exercise 8

In this exercise we are looking at the golden rule capital stock. This is the capital stock that maximizes steady state consumption. If we, as we did at the seminar (and in 6b), assume zero capital depreciation $\delta = 0$, steady state consumption is given from the resource constraint with $k_{t+1} = k_t$ as

$$c + g = k^\alpha$$

We see that, absent capital depreciation, steady state consumption is increasing in the capital stock, hence the capital stock that maximizes SS consumption is $k = \infty$. To get a finite level of golden rule capital stock, we therefore assume $\delta > 0$, giving the resource constraint

$$c_t + g_t + k_{t+1} = k_t^\alpha + (1-\delta)k_t$$

which in steady state ($k_{t+1} = k_t$) simplifies to

$$c + g = k^\alpha - \delta k$$

Steady state wage and real interest rate.

From the consumption Euler equation we get that

$$c_{t+1} = [\beta(1 + r_{t+1})]^{\frac{1}{\theta}} c_t$$

Steady state requires constant consumption, reducing the Euler equation to

$$\begin{aligned} \beta(1 + r_{t+1}) &= 1 \\ r_{t+1} &= r_{ss} = \frac{1}{\beta} - 1 \end{aligned}$$

[2] In other words, as long as the initial capital is the same, the maximization problems starting in period 0 and period s are identical.

with $\delta > 0$ the real interest rate equals the marginal product of capital net of depreciation, hence the steady state capital stock is given by

$$\begin{aligned} r_{ss} &= \alpha k_{ss}^{\alpha-1} - \delta = \frac{1}{\beta} - 1 \Rightarrow \\ k_{ss} &= \left(\frac{1}{\alpha} (\frac{1}{\beta} + 1 - \delta) \right)^{\frac{1}{\alpha - 1}} \end{aligned}$$

Finally, the steady state wage rate is given by the marginal product of labor

$$w_{ss} = (1 - \alpha) k_{ss}^{\alpha}$$

Golden rule capital stock and real interest rate

The golden rule capital stock k_{gold} maximizes steady state consumption

$$c + g = k^{\alpha} - \delta k$$

Taking the FOC wrt the right hand side gives

$$\begin{aligned} \alpha k_{gold}^{\alpha-1} - \delta &= 0 \Rightarrow \\ k_{gold} &= \left(\frac{\delta}{\alpha} \right)^{\frac{1}{\alpha - 1}} \end{aligned}$$

This means that

$$\begin{aligned} r_{ss} &= \alpha k_{ss}^{\alpha-1} - \delta = \frac{1}{\beta} - 1 \\ r_{gold} &= \alpha k_{gold}^{\alpha-1} - \delta = 0 \end{aligned}$$

Because of discounting ($\beta < 1$), the real interest rate under the golden rule is lower than the steady state interest rate in the Ramsey growth model, implying that the capital stock is larger $k_{gold} > k_{ss}$. Why? The intuition is that in the Ramsey growth model, the objective is not to maximize utility in steady state. The representative agent also cares about consumption during the transition to steady state. The agent could, by reducing consumption the short run, move from k_{ss} to $k_{gold.}$ However, because of discounting, the value of the eventual permanent increase in consumption is bounded, and the short term loss will at some point outweigh the permanent long-run gain. Recalling the phase diagram, this happens when k is larger than k_{ss}. The saddle path, describing the *optimal* trade-off between the short and long run, implies that it is optimal to reduce the capital stock whenever $k > k_{ss}$.

Problem set 2, Debt crisis

2*a. The equilibria of the model*

The equilibria of the model occurs when both equilibrium conditions are met. That is, given a perceived probability of default $\pi^* < 1$, investors are willing to hold the debt if the interest rate is $R^* = \overline{R}(1-\pi^*)$. Given the interest rate R^* the actual default probability is given by $\pi^* = F(R^*D)$. Graphically, the equilibria occurs as the intersection between the two curves

$$\begin{aligned} \pi &= 1 - \frac{\overline{R}}{R} \\ \pi &= F(RD) \end{aligned}$$

Drawing the curves as in Romer (chapter 12), gives two intersections. One with a low probability of default and interest rate (π_A, R_A) (the normal state) and one with a high probability of default and interest rate (π_B, R_B) (B the unstable equilibrium). There is also a third equilibrium (C) the crisis state, when the government defaults immediately. This occurs when there is no interest rate R that will induce investors to hold government debt, which will happen if investors believe that the government defaults with certainty the next period. This means that the government is unable to roll over debt, and thus defaults today.

2*b. stability of the normal and crisis state*

Suppose that we are in the normal equilibrium (A). If investors suddenly increase their estimate of default probability from π_A to π^1 they require a higher interest rate $R^1 = (1-\pi^1)\overline{R} > R_A$ to hold the debt (think of a small shift, so that we still have $R^1 < R_B$) But at this interest rate, the actual probability of default (π^{act}) is lower than the investors estimate, $\pi^{act} = F(R^1D) < \pi^1$ (the $F(RD)$ curve is below the curve $1 - \frac{\overline{R}}{R}$ at $R = R^1$). It is reasonable to believe that rational investors over time will adjust their estimated default probability down to $\pi^2 = \pi^{act}$. The interest rate required to roll over debt is then $R^2 = (1-\pi^2)\overline{R} < R^1$. But still the actual probability of default is below the what the investors believe ($\pi^{act} = F(R^2D) < \pi^2$) and they adjust their estimate further down. This process goes on until we converge back to the equilibrium in A.

Suppose that we are in the crisis state in which investors are unwilling to hold D. If investors suddenly believe that the probability of default is $\pi^1 < 1$, they will hold D if the interest rate is $R^1 = (1-\pi^1)\overline{R}$ (think of a shift such that have $R_B < R^1 < R_U$). At this interest rate, the actual probability of default is higher than what the investors believe $\pi^{act} = F(R^1D) > \pi^1$,(the $F(RD)$ curve is above the curve $1 - \frac{\overline{R}}{R}$ at $R = R^1$). Investors then adjust their probability of default upwards, and the process continues until the actual probability of default is equal to 1 and the government is unable to roll over debt.

2*c and d. effects of shifts in* $\overline{R}$ *and* D

If the risk free return increases, nothing will happen to the curve $\pi = F(RD)$. However, the curve $\pi = 1 - \frac{\overline{R}}{R}$ will shift down and to the right, moving equilibrium A to the right and equilibrium B to the left. For any given probability of default, it now requires a higher return on government debt for investors to hold it. Hence, the normal and the unstable equilibrium move closer. If the risk free return becomes large enough, the curves will never intersect, leaving the crisis state as the only equilibrium.

If government debt increases, nothing will happen to the curve $\pi = 1 - \frac{\overline{R}}{R}$. However, the probability of default will increase for any given level of R, moving $F(RD)$ up and to the left. As above, the two equilibria move closer together and If the increase in D is large enough, the two curves will never intersect

Problem set 2, Life cycle behavior, exercise 2

This exercise is about deriving the evolution of the capital stock in the OLG model. The evolution of capital is always defined as

$$k_{t+1} = k_t(1-\delta) + I_t$$

where the next period capital stock is current capital stock after depreciation plus investment in new capital I_t. We assumed $\delta = 0$ (doesn't matter for the argument)[3], so

$$k_{t+1} = k_t + I_t$$

2a)

At the Monday and Tuesday seminar we used that period t investment is the difference between period t production and consumption, and got

$$k_{t+1} = k_t + y_t - c_t^y - c_t^o$$

Inserting for consumption by the young, and old generation, and using the expression for wage and interest rate[4] we found that

$$k_{t+1} = k_t + y_t + s_t^y - w_t - (1+r_t)k_t = s_t^y + y_t - w_t - r_t k_t = s_t^y + y_t - y_t = s_t^y$$

where the first equality follows from the definition of saving by the young generation $s_t^y = w_t - c_t^y$ and the old generation's budget constraint $c_t^o = (1+r_t)b_t^o$. Note that since the young generation does not have any financial asset, aggregate financial assets in the economy is held by the old generation $b_t = b_t^o$. Implicitly, they are the owner of the entire capital stock. Formally this follows from the capital market clearing condition $k_t = b_t = b_t^o$. Consequently, $c_t^o = (1+r_t)k_t$.

At the Thursday seminar we used an alternative (but equivalent) approach, where we utilized the fact that investment in period t equals savings in period $t : I_t = s_t^y + s_t^o$

$$k_{t+1} = k_t + I_t = k_t + s_t^y + s_t^o$$

Saving is defined as income minus consumption. The old generation does not work, so income consists of asset return $r_t b_t$ and we get, $s_t^o = r_t b_t - c_t^o = r_t b_t - (1+r_t)b_t^o = -b_t^o = -k_t^o$ and

$$k_{t+1} = k_t + s_t^y - k_t = s_t^y$$

[3] Actually, in a two period OLG model it is more common to assume full capital depreciation $\delta = 1$, since we think of a period consisting of many years (e.g. 30 years)

[4] Note that there is a typo in the problem set where we define the real interest rate. It should say $r_t = f'(k_t)$.

2b and c

Now that we know that the evolution of capital is determined by the saving decision of the young generation, the next step is to insert for the optimal savings decision from the life-cycle problem of the young. Using the result in exercise 1 we know that the young consumes a fraction $\frac{1}{1+\beta}$ of life time income. Life-time income consists of wage income when young plus the discounted labor income when old. Since they don't work when old, and they are young in period t the life time income is simply w_t. It then follows that

$$s_t^y = w_t - c_t^y = w_t - \frac{1}{1+\beta}w_t = \frac{\beta}{1+\beta}w_t$$

inserting for the wage rate we get that the saving decision is completely determined by the current capital stock (and exogenous preference and technology parameters)

$$s_t^y(k_t) = \frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha$$

and the evolution of capital is

$$k_{t+1} = s_t^y(k_t)$$

The steady state capital stock ($k_{t+1} = k_t = k_{ss}$) is the unique solution to the equation

$$k_{ss} = s_t^y(k_{ss})$$

This says that in steady state, the current capital stock gives the wage rate that induces the young to save an amount equal to the current capital stock