1 Growth and business cycles (80%)

1. Definition of a competitive equilibrium: A competitive equilibrium is an allocation $\{c_t, h_t, a_t, K_t\}$

   - Households maximize utility subject to the budget constraint, taking the wage w and interest rate r as given.
   - Firms maximize profits, taking the wage, the interest rate and product price (normalized to unity) as given. Due to the constant return to scale, this implies that profits are zero and prices equal marginal productivities:

   $$\begin{aligned} w_t &= z_t (1-\alpha) (K_t)^{\alpha} (z_t H_t)^{-\alpha} \\ r_t &= w_t = \alpha (K_t)^{\alpha-1} (z_t H_t)^{1-\alpha} - \delta \end{aligned}$$

   - All markets (for capital, labor and goods) clear

   $$\begin{aligned} h_t &= H_t \\ a_t &= K_t \\ Y_t &= C_t + K_{t+1} - (1-\delta) K_t \end{aligned}$$

2. Lagrangian is

   $$L = \sum_{t=0}^{\infty} \left(\beta^t [\log c_t - \frac{1}{2} h_t^2] - \lambda_t [c_t + a_{t+1} - (1+r_t) a_t - w_t h_t] \right)$$

   First-order conditions:

   $$\begin{aligned} c_t &: \quad \beta^t \frac{1}{c_t} = \lambda_t \\ h_t &: \quad \beta^t h_t = \lambda_t w_t \\ a_{t+1} &: \quad (1 + r_{t+1}) \lambda_{t+1} = \lambda_t \end{aligned}$$

   Combine foc for c_t and a_{t+1} to obtain the Euler equation

   $$\frac{1}{c_t} = \beta (1 + r_{t+1}) \frac{1}{c_{t+1}}$$

   while foc for c_t and h_t give the intratemporal optimality condition:

   $$h_t = \frac{w_t}{c_t}$$

   Students should also provide an interpretation of these conditions.

3. Why could we obtain the comp. equilibrium by solving a social planner problem? Due to the first welfare theorm, any competitive equilbrium is Pareto optimal. Moreover, any Pareto optimal allocation is the solution to a planner problem with some weights. More precisely, there exists a set of planner weights (on different people in the economy) such that the planner would prefer that particular Pareto optimal allocation to any other feasible allocation. Finally, due to the second welfare theorem, any pareto optimal allocation can be supported by a competitive equilibrium with transfers. We conclude that instead of solving for the prices and allocations of the competitive equilibrium, we could solve directly for the pareto optimal allocation that the social planner would choose, suject to the feasibility constraint (and given some weights). It is obvious that for the representative agent economy, the relevant planner weights are to have identical weights on all people.

4. Steady state.

   (a) A steady state is a situation where output, consumption and capital per unit of efficient labor, zh, is constant. The wage rate must equal the marginal product of labor,

   $$w_t = (1-\alpha)K_t^\alpha (z_t H_t)^{1-\alpha}\frac{1}{H_t}$$

   Writing it in terms of capital per efficient unit we get:

   $$w_t = (1-\alpha)z_t \left(\frac{K_t}{z_t H_t}\right)^\alpha$$

   Hence if K/zH is constant, the wage rate will grow at the same rate as z_t, namely g.

   (b) Use the Euler equation to write

   $$\frac{\tilde{c}_{t+1}}{\tilde{c}_t}\frac{H_{t+1}z_{t+1}}{H_t z_t} = \beta(1+r_{t+1})$$

   where $\tilde{c} = c/zH$. Since consumption per unit of efficient labor by assumption must be constant in steady state, the ratio of consumption-terms is equal to one. Letting g_e be the growth rate of efficient labor, we see that the steady state interest rate is positive for two reasons: (i) discounting ($\beta < 1$) and growth ($g_e > 0$).

(c) To answer this, we look at the intratemporal optimality condition:

$$h_t = \frac{w_t}{c_t}$$

We have already figured out that w grows at a rate g, while c grows at a rate g_e. The latter may differ from g only if h is non-constant in steady state. Assume that h grows over time. This makes $g_e > g$, but that would make the RHS of the equation grow while the LHS falls over time. Hence not possible. Cannot have h falling over time either, for the same reason. Only possible steady state growth rate is a constant value of h and $g_e = g$.

5. To do this calibration, students should use the Euler equation, evaluate it for the steady state from question 4, and find out how to calibrate β to make the interest rate equal 0.01 in steady state. The steady state Euler equation can be transformed to

$$\beta = \frac{1+g}{1+r}$$

where g_c is the steady state growth rate. At this point students must remember that consumption grows with z. Some students may forget that consumption is growing (since the calibration exercise in class assumed no long-run growth), and therefore answer $\beta = 1/1.01$. This answer should also be given some (but not full) credit.

6. The difference between models with divisible and indivisible labor has been discussed on several occasions in class. In a model with divisible labor, the labor supply decision follows from an intratemporal condition similar to the one derived in this problem set. For a model with indivisible labor and labor lotteries, we get a model *as if* the representative agent's disutility of labor is linear. This implies an infinite Frisch elasticity in the aggregate, no matter what it is for each individual (top-students may note that the Frisch elasticity in the model for this problem set is 1/2). The result is that a model with indivisible labor and labor lotteries will, all else equal, produce much larger labor supply response to productivity shocks, and therefore also greater internal propagation of shocks. This means that Figure 1 must plot the impulse-response for the model with *indivisible* labor, while Figure 2 is the plot for a model for *divisible* labor.

7. The students are used to look at the steady state for growth models without labor supply. In that case they look at the Euler equation

$$\frac{c_{t+1}}{c_t} = \beta(1 + r_{t+1}) = \beta\left(1 + \alpha\left(\frac{K_t}{H_t}\right)^{\alpha-1} - \delta\right) \tag{1}$$

and the aggregate resource constraint

$$C_t + K_{t+1} = K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t. \tag{2}$$

where z_t is set to one for simplicity (because it's assumed to be constant). The aim is to impose constant consumption in equation (1) and constant capital in equation (2) to find two curves governing the phase diagram dynamics. The saddle path is found in the south-west, north-east direction in a k, c diagram.

The real difficulty in this problem is that labor is endogenous. We propose two ways to grade this exercise:

(a) Suppose the student ignores that labor supply is endogenous. With a constant H, the phase diagram is as in the standard growth model without labor supply, which we studied in class. If that phase diagram is done ok, the student should get some credit

(b) A very ambitious studen should also take into account that labor supply is endogenous. The trick is to use the intratemporal condition to substitute out h from the above equations. The intratemporal condition is:

$$h_t = \frac{w_t}{c_t}.$$

Combine this with an expression for the marginal product of labor as the wage rate ($w_t = (1-\alpha)z_t\left(\frac{K_t}{z_t H_t}\right)^{\alpha}$) to obtain an expression for h_t:

$$\begin{aligned} H_t &= \frac{w_t}{C_t} = \frac{(1-\alpha)\left(\frac{K_t}{H_t}\right)^{\alpha}}{C_t} \\ &\Rightarrow \\ H_t &= (1-\alpha)\left(\frac{1}{C_t}\right)^{\frac{1}{1+\alpha}} (K_t)^{\frac{\alpha}{1+\alpha}} \end{aligned}$$

The resource constraint (2) then becomes

$$\begin{aligned}
C_t + K_{t+1} &= K_t^\alpha \left((1-\alpha) \left(\frac{1}{C_t}\right)^{\frac{1}{1+\alpha}} (K_t)^{\frac{\alpha}{1+\alpha}} \right)^{1-\alpha} + (1-\delta)K_t \\
&= (1-\alpha)^{1-\alpha} \left(\frac{1}{C_t}\right)^{\frac{1-\alpha}{1+\alpha}} (K_t)^{\frac{2\alpha}{\alpha+1}} + (1-\delta)K_t
\end{aligned}$$

Suppose that consumption is such that the capital stock remains constant, i.e., $K_{t+1} = K_t = K$, so

$$C_t = (1-\alpha)^{1-\alpha} \left(\frac{1}{C_t}\right)^{\frac{1-\alpha}{1+\alpha}} (K_t)^{\frac{2\alpha}{\alpha+1}} - \delta K_t$$

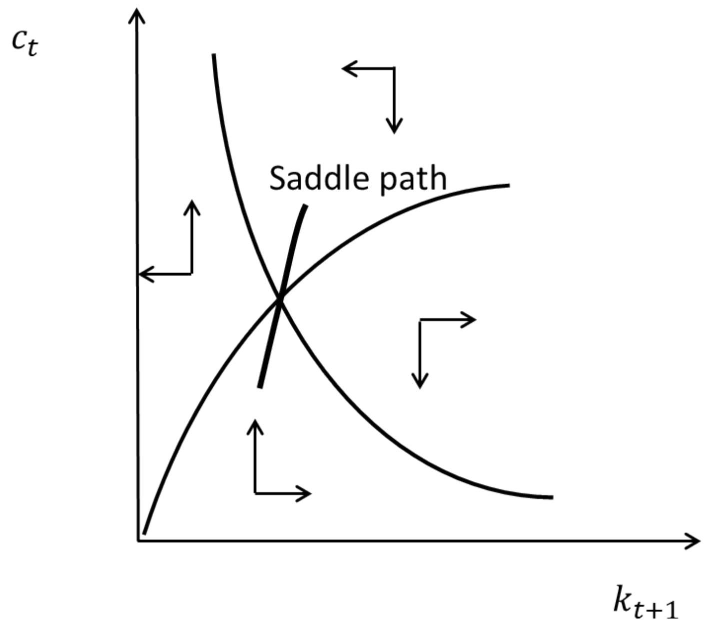
In this equation, if $K = 0$, then consumption must be zero $C = 0$, so the graph must start at origo. The graph is hump-shaped. For (C, K) combinations above the graph, capital will decrease, while capital will increase below the graph.

Consider now the Euler equation when consumption is constant:

$$\begin{aligned}
1 &= \beta \left(1 + \alpha \left(\frac{K_t}{H_t}\right)^{\alpha-1} - \delta \right) \\
&= \beta \left(1 + \alpha \left(\frac{K_t}{(1-\alpha)\left(\frac{1}{C_t}\right)^{\frac{1}{1+\alpha}} (K_t)^{\frac{\alpha}{1+\alpha}}} \right)^{\alpha-1} - \delta \right) \\
&\Rightarrow \\
\left(\frac{\frac{1}{\beta} + \delta - 1}{\alpha} \right)^{\frac{1}{1-\alpha}} &= (1-\alpha) \left(\frac{1}{C_t}\right)^{\frac{1}{1+\alpha}} (K_t)^{-\frac{1}{\alpha+1}} \\
C_t &= (1-\alpha) \left(\frac{\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{1+\alpha}{1-\alpha}} \frac{1}{K_t}
\end{aligned}$$

This graph defines the border. For (C, K) combinations to the left of this graph, consumption will be on an increasing path, since the interest rate is higher than the discount rate. Conversely, consuimption will be on a decreasing path to the right of this graph (because the interest rate is low).

Combine the two curves in a phase diagram. The saddle path is found in the south-west, north-east direction in a k, c diagram.



2 Debt crisis (20%)

This problem tests their understanding of the 'debt crisis' model gone through in class (lecture 7), as well as in Romer's textbook (see chapter on fiscal policy). As is stated in the questions, students are expected to provide intitution, no math, but students are likely to put up some equations nevertheless. An A-answer should include the following:

- When analyzing the market for government debt under default risk, (at least) two aspects should be in focus. First, what is investor's required return on a government's debt for a given (perceived) probability of default? Second, what is the *actual* probability of default for a given return on the debt?

- Investors' required return depends on what the risk-free (if any) alternative is. If investors are risk netural, there is a simple increasing and concave relationship between the required return and probability of default.

- The probability of default will depend on the interest the government must pay on its debt, since that makes any given stock of debt more expensive to service. Imagine that the government repays its debt if future tax income is sufficiently high. If tax revenues are too low, it defaults on the entire debt. For a given distribution of future taxes, this gives an exact mapping between the interest rate on the government debt and probability of default. It is likely to be an increasing function (higher interest rate will make default more likely), but it can for instance take an S-shape (which happens if future taxe income has a symmetric distribution).

- An equilibrium requires investors perceived risk to be in accordance with the actual probabilty of default. At this point it will be useful to draw a figure as the one below to illustrate the situation.

- The model in question turns out to have multiple equilibria. Assuming an S-shaped relationship between the actual probability of default and interest rate, we get *three* equilibria.

- One equilibrium has a low interest rate and a low probability of default. It is likely to be *stable*, since a small change in investors' perceived probability of default will not be self-fulfilling.

- Another equilibrium implies a complete break-down of the debt market. This is the case where the probability of default approaches 1, while the equilibrium interest goes to infinity. This is also stable, since a small reduction in investor's percieved probabilty of default is not self-fulfilling.

- The third equlibrium is an unstable 'tipping point' with a moderate interest rate and moderate risk of default. It is unstable, and a small variation in the perceived probability of default will send us to one of the two stable equilibria.

- A debt crisis can be interpreted as ending up in the break down equilibrium.

- The model has two main points that the students should emphasize (this is also given to them as a hint in the question). First, the model helps us understand how *fundamentals* matter for the likelihood of a debt crisis. A larger stock of debt, or worsened outlooks for future tax revenues, will make the probability of default higher for any level of interest. This makes the 'good' equilibrium worse (higher interest rate and higher prob of default), and moves the tipping point closer to the good equilibrium (making a complete debt crisis more likely). Second, a debt-crisis can also be initiated by self-fulfilling prophesies (speculative attacks). That is because of the multiplicity. No matter how sound the fundamentals are, there is always a chance that investors perceived probability of default shifts up sufficiently much to push an economy from a 'good' to a 'crisis' equilibrium.

- Conclusion to the Minister? Greeks may complain to some degree about speculators, but their own fundamentals are making them more vulnerable.