Neoclassical growth model

Econ 4310 Lecture 3

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Consumers

$$\max U_0 = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + ... = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

s.t.

$$c_t = (1 + r_t)a_t + w_t - a_{t+1} \qquad t = 0, 1, 2, \dots \tag{2}$$

$$\lim_{t\to\infty} a_t R_{t-1}^{-1} \geq 0 \qquad R_t = \prod_{s=0}^{t} (1 + r_s) \tag{3}$$

$$a_0 \text{ given}$$

a_t = net assets carried over from period $t - 1$ to period t, r_t = interest rate paid on these

Budget constraint, finite horizon

Constant interest rate, two periods:

$$
\begin{aligned}
a_1 &= a_0(1+r) + w_0 - c_0 \\
a_2 &= (1+r)a_1 + w_1 - c_1 \\
a_2 &= (1+r)^2 a_0 + (1+r)[w_0 - c_0] + [w_1 - c_1]
\end{aligned}
$$

Budget constraint: $a_2 \geq 0$. Equivalent formulation:

$$
a_2(1+r)^{-1} = a_0(1+r) + [w_1 - c_1] + (1+r)^{-1}[w_2 - c_2] \geq 0
$$

Budget constraint with infinite horizon

$\lim_{t\to\infty} a_t \geq 0$ Debt must be paid back!

$\lim_{t\to\infty}(1+r)^{-(t-1)}a_t \geq 0$ Debt can be rolled over forever!Interest must be paid from earned income. No Ponzi-Game
Constant versus variable r:

$$R_{t-1} = (1+r)^{t-1} \text{ vs } R_{t-1} = (1+r_0)(1+r_1)\cdots(1+r_{-t-1}) = \prod_{s=0}^{t-1}(1+r_s)$$

Budget constraint continued

$$\text{NPG} \quad \lim_{t\to\infty} (1+r)^{-(t-1)} a_t \geq 0 \tag{4}$$

- If $a_t \geq 0$ for all $t \geq$ some t', NPG is always satisfied
- If $a_t < 0$ for all $t \geq$ some t', NPG can be satisfied only if the rate of increase in the debt is below the interest rate:

$$\frac{|a_t|}{|a_{t-1}|} < 1 + r \Leftrightarrow a_t > (1+r)a_{t-1} \tag{5}$$

- The last inequality requires that some interest is paid from current earnings, not by taking on new debt.
- Condition (5) is necessary, but not sufficient for (4) to hold when $a < 0$. The amount that is paid need to keep pace with size of debt.

More on constraint

- A sufficient condition is that a constant share of the interest payments are paid out of current earnings. However, when this share is below 100 per cent, the amount paid from current earnings goes to infinity as the deb increases.
- If there is no trend growth, NPG can only be satisfied if in the limit *all* interest is paid from current earnings.
- In the present model this limits household borrowing to w/r.
- NPG is always satisfied if a constant debt is rolled over forever:
- A constant debt requires that all interest is paid from current income or $c_t = w_t - rb$ where b is the size of the debt.

Present value version

Assets accumulated at the end of period $t-1$:

$$a_t = a_0(1+r)^t + \sum_{j=0}^{t-1}(w_j - c_j)(1+r)^{t-1-j}$$

Present value of a_t is:

$$a_(1+r)^{-(t-1)} = a_0(1+r) + \sum_{j=0}^{t-1}(w_j - c_j)(1+r)^{-j}$$

Taking limits on both sides, we get

$$\lim_{t\to\infty}(1+r)^{-(t-1)}a_t = a_0(1+r) + \sum_{j=0}^{\infty}(w_j - c_j)(1+r)^{-j} \geq 0$$

or

$$\sum_{j=0}^{\infty} c_j(1+r)^{-j} \leq a_0(1+r)\sum_{j=0}^{\infty} w_j(1+r)^{-j} \qquad (6)$$

First-order conditions

Use budget equation to substitute for c in utility:

$$U_0 = \ldots \beta^t u((1+r_t)a_t + w_t - a_{t+1}) + \beta^{t+1} u((1+r_{t+1})a_{t+1} + w_{t+1} - a_{t+2}) + \ldots$$

Differentiate with respect to a_{t+1}

$$\frac{\partial U_0}{\partial a_{t+1}} = \beta^t u'(c_t)(-1) + \beta^{t+1} u'(c_{t+1})(1 + r_{t+1}) = 0$$

Consumption Euler equation

$$u'(c_t) = \beta u'(c_{t+1})(1 + r_{t+1}) \qquad t = 1, 2, \ldots \tag{7}$$

Terminal condition

Budget constraint should be satisfied with equality

$$\lim_{t \to \infty} a_t R_{t-1}^{-1} = 0 \qquad (8)$$

- If positive, assets should not grow faster than interest rate
- A part of interest income should be spent

Demand functions

Solution to optimization problem has to satisfy

- First order conditions
- Period by period budget equations
- Present value budget constraint with =

Demand functions

$$C_t = D_t(W_0, r_1, r_2, r_3, \ldots) \tag{9}$$

- Consumption depends on total wealth, not on income each year
- The distribution of consumption on periods is independent of the distribution of income

Labor and capital

$$f'(k_t) = r_t \tag{10}$$

$$f(k_t) - k_t f'(k_t) = w_t \tag{11}$$

Equilibrium conditions

$$a_t = k_t \tag{12}$$

$$c_t + k_{t+1} = k_t + f(k_t) \tag{13}$$

$$k_t \geq 0 \quad t = 1, 2, 3 \ldots, \infty \tag{14}$$

Relation to planner's optimum

From combining Euler and $r = f'(k)$:

$$u'(c_t) = \beta u'(c_{t+1})(1 + f'(k_{t+1})) \tag{15}$$

- Same difference equations, same stationary point, same phase diagram
- Paths where $k \to 0$, violate the budget constraint
- Paths where $k \to \infty$ do not use the whole budget; as r goes to zero consumption stays below wage income forever

Competitive markets, Pareto-efficiency and welfare optimum