Questions & Answers

ECON4310, University of Oslo

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Problem Set 0

- **What is the difference between the rental rate and the interest rate?**
- **Exercise 0.1 (f): Doesn't the rental rate fall in the transition to the steady-state?**
- **Exercise 0.2 (a): What is the correct expression for the optimal consumption level?**

Problem Set 1

- **Exercise 1.1 (a): Could we also write the Lagrangian as**

$$\tilde{\mathcal{L}} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left(w_t + (1 + r_t)a_t - \tau_t - c_t - a_{t+1} \right) \right],$$

  such that the Lagrange multiplier is effectively $\beta^t \lambda_t$ instead of λ_t?

- **Exercise 1.1 (g): Shouldn't the goods market clearing condition be**

$$c_t + G_t + (k_{t+1} - k_t) = y_t.$$

Problem Set 2

- **What is the difference between using the net present value budget constraint and the period-by-period budget constraint of the household when solving the consumer problem? How do I know when to use which formulation?**

Problem Set 3

- **Exercise 3.1 (a): Shouldn't the optimal savings be**

$$s = \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta - 1}} \left(\beta^{1/\theta}(1+r)^{1/\theta - 1} w_1 - \frac{w_2}{1+r} \right).$$

- **How does the equilibrium of the overlapping generations model look like if there is no productive capital to save in?**

Problem Set 4

- **Exercise 4.1 (a): Why does the Ricardian equivalence proposition not apply to this economy? How could it be restored?**

Problem Set 6

- **Exercise 6.1 (a): Could I also use the probability weighted terms λ_0, $\beta p \lambda_1(s_G)$, and $\beta(1-p)\lambda_1(s_B)$ as the Lagrange multipliers on the state-by-state constraints?**

- **Exercise 6.1 (d): Can you put straight the confusion about Jensen's inequality?**

- **Exercise 6.1 (f): Here is the correct derivation of the implicit function's derivative that I messed up in the last 8-10 Monday seminar.**

Problem Set 7

- **Exercise 7.A.1: Does the absence of the precautionary savings motive imply that there are no savings?**

Problem Set 0

Q: *What is the difference between the rental rate and the interest rate?*

A: The **rental rate** is equal to the marginal product of capital, $\partial Y_t/\partial K_t$, and the relevant price for the firm when it decides how much capital intensive to produce. Remember that firms make zero profits, so firms do not own any capital (they have no profits to buy any!), but have to rent it from investors.

The **interest rate** is equal to an investor's return on renting out capital to the firm: the investor rents out K_t to the firm, after production the firm returns the capital K_t plus the rent $\partial Y_t/\partial K_t \times K_t$ to the investor, and on top of that a fraction δK_t of capital depreciates. The net return for the investor is therefore

$$\frac{(1-\delta)K_t + \partial Y_t/\partial K_t \times K_t}{K_t} - 1 = \partial Y_t/\partial K_t - \delta.$$

Thus, the **interest rate is equal to the rental rate minus the depreciation rate**.

Q: *Exercise 0.1 (f): Doesn't the rental rate fall in the transition to the steady-state?*

A: Yes it does, and I think I stated this wrongly in Seminar 4. Here's the correct explanation: the growth rate of the rental rate is given by

$$\frac{r_{t+1}}{r_t} - 1 = (k_{t+1}/k_t)^{\alpha-1} - 1.$$

Remember that $k_{t+1}/k_t > 1$ in the transition. However, as the exponent $\alpha - 1$ is negative, the term $(k_{t+1}/k_t)^{\alpha-1}$ will be smaller than 1 during the transition such that after the initial jump due to immigration the rental rate is falling on the transition back to the steady-state level.

Q: *Exercise 0.2 (a): What is the correct expression for the optimal consumption level?*

A: I think I messed up writing the correct exponent in at least Seminar 2. The correct expression is

$$c = (\psi\theta)^{1/(1-\theta)} \left[\frac{1}{(1-\tau^n)w}\right]^{\theta/(1-\theta)} = (\psi\theta)^{1/(1-\theta)} \left[(1-\tau^n)w\right]^{-\theta/(1-\theta)}.$$

Problem Set 1

Q: *Exercise 1.1 (a): Could we also write the Lagrangian as*

$$\tilde{\mathcal{L}} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left(w_t + (1+r_t)a_t - \tau_t - c_t - a_{t+1} \right) \right],$$

instead of

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t u(c_t) + \lambda_t \left(w_t + (1+r_t)a_t - \tau_t - c_t - a_{t+1} \right) \right].$$

A: Yes, we could! That doesn't change the consumption Euler equation, but only the interpretation of λ_t (which is no longer the Lagrange multiplier, but $\beta^t \lambda_t$ is). Interestingly, this transformation makes λ_t independent of time t conditional on the consumption level. To see this, look at the optimality condition with respect to c_t in the case of the first formulation

$$0 = \frac{\partial \tilde{\mathcal{L}}}{\partial c_t} = \beta^t u'(c_t) - \beta^t \lambda_t \quad \Leftrightarrow \quad \lambda_t \equiv f(c_t) = u'(c_t),$$

and in the case of the second formulation

$$0 = \frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t \quad \Leftrightarrow \quad \lambda_t \equiv f(t, c_t) = \beta^t u'(c_t).$$

Thus, the first formulation of the Lagrangian is more convenient if you want λ_t to be a "time-homogeneous" function f which is independent of calendar time (as is the case in recursive macroeconomics).

Q: *Exercise 1.1 (g): Shouldn't the goods market clearing condition be*

$$c_t + G_t + (k_{t+1} - k_t) = y_t.$$

A: Yes, it should. I messed this up in the Monday seminars, and here's the explanation: combine the private budget constraint of the agent

$$c_t + (a_{t+1} - a_t) = w_t + r_t a_t - \tau_t$$

with the government's budget constraint

$$\tau_t = G_t - (D_{t+1} - D_t) + r_t D_t$$

to yield

$$\begin{aligned}c_t + (a_{t+1} - a_t) &= w_t + r_t a_t - [G_t - (D_{t+1} - D_t) + r_t D_t] \\ &= w_t + r_t a_t - G_t + (D_{t+1} - D_t) - r_t D_t.\end{aligned}$$

Rewrite this equality as

$$c_t + G_t + [(a_{t+1} - D_{t+1}) - (a_t - D_t)] = w_t + r_t(a_t - D_t).$$

Market clearing in the capital market implies that $a_t - D_t = k_t$, such that you can rewrite the above equation as

$$\begin{aligned}c_t + G_t + (k_{t+1} - k_t) &= w_t + r_t k_t \\ &= y_t,\end{aligned}$$

where the last step follows from the fact that firms make zero profits $0 = \pi_t = y_t - (w_t + r_t k_t)$.

Problem Set 2

Q: *What is the difference between using the net present value budget constraint and the period-by-period budget constraint of the household when solving the consumer problem? How do I know when to use which formulation?*

A: In the finite horizon problem the consumer maximizes utility subject to the period-by-period budget constraint

$$a_{t+1} = (1 + r_t)a_t + w_t - c_t, \quad \forall t, \tag{1}$$

and the terminal condition

$$a_{T+1} = 0 \tag{2}$$

(we assume equalities here from the beginning for the ease of exposition). The period-by-period budget constraints can therefore be reduced (by substituting out all endogenous variables a_{t+1}, for all $t < T$) to the single constraint in net present value terms

$$\sum_{t=0}^{T} \frac{c_t}{\Pi_{s=0}^{t}(1 + r_s)} = a_0 + \sum_{t=0}^{T} \frac{w_t}{\Pi_{s=0}^{t}(1 + r_s)} - \frac{a_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)}.$$

If impose on top of that the terminal condition, we can also get rid of a_{T+1} and end up with the consumers lifetime budget constraint in net present value terms

$$\sum_{t=0}^{T} \frac{c_t}{\Pi_{s=0}^{t}(1 + r_s)} = a_0 + \sum_{t=0}^{T} \frac{w_t}{\Pi_{s=0}^{t}(1 + r_s)}. \tag{3}$$

Thus, maximizing utility subject to Equations (1) and (2) is equivalent to maximizing utility subject to Equation (3) only. If possible, I would always go for the formulation with the lifetime budget constraint as you only have to keep track of one Lagrange multiplier instead of $T + 1$ Lagrange multipliers.

In the infinite horizon problem the consumer maximizes utility subject to the same period-by-period budget constraint

$$a_{t+1} = (1 + r_t)a_t + w_t - c_t, \quad \forall t, \tag{4}$$

and the no-Ponzi condition

$$\lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0. \tag{5}$$

The period-by-period budget constraints can be reduced to the single constraint in net present value terms

$$\sum_{t=0}^{\infty} \frac{c_t}{\Pi_{s=0}^{t}(1+r_s)} = a_0 + \sum_{t=0}^{\infty} \frac{w_t}{\Pi_{s=0}^{t}(1+r_s)} - \lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{s=0}^{T}(1+r_s)}.$$

If impose on top of that the no-Ponzi condition, we can also get rid of the last term and end up with the consumers lifetime budget constraint in net present value terms

$$\sum_{t=0}^{\infty} \frac{c_t}{\Pi_{s=0}^{t}(1+r_s)} = a_0 + \sum_{t=0}^{\infty} \frac{w_t}{\Pi_{s=0}^{t}(1+r_s)}. \tag{6}$$

Thus, maximizing utility subject to Equations (4) and (5) is equivalent to maximizing utility subject to Equation (6) only. If possible, I would always go for the formulation with the lifetime budget constraint as you only have to keep track of one Lagrange multiplier and you can ignore the so called transversality condition

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0, \tag{7}$$

which pops up as an additional optimality condition if maximizing utility subject to Equations (4) and (5).

How do you know when to use which formulation? If possible, I would always work with the formulation of budget constraint in net present value terms. BUT, sometimes this is not possible. Consider for example the resource constraint of the planner problem

$$k_{t+1} = k_t^{\alpha} + (1-\delta)k_t - c_t, \quad \forall t,$$

which is also subject to the non-negativity constraint on physical capital $k_{t+1} \geq 0$, for all t. In this case it is much simpler to just go with the period-by-period constraint formulation, as solving for the lifetime constraint yields a very complicated expression.

Problem Set 3

Q: *Exercise 3.1 (a): Shouldn't the optimal savings be*

$$s = \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \left(\beta^{1/\theta}(1+r)^{1/\theta-1} w_1 - \frac{w_2}{1+r} \right).$$

A: Yes, there should be a minus instead of a plus in front of $w_2/(1+r)$. I messed this up in the early Monday seminar, and here's the correct derivation

$$\begin{aligned} s &= w_1 - c_1 \\ &= w_1 \left(\frac{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} - \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \right) \\ &\quad - \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \frac{w_2}{1+r} \\ &= \frac{1}{1 + \beta^{1/\theta}(1+r)^{1/\theta-1}} \left(\beta^{1/\theta}(1+r)^{1/\theta-1} w_1 - \frac{w_2}{1+r} \right). \end{aligned}$$

Q: *How does the equilibrium of the overlapping generations model look like if there is no productive capital to save in?*

A: Suppose that the consumption good can be stored across periods (otherwise, if the good is perishable, than the agent will simply consume the endowment in each period as there is no storage or savings technology). Then the agent will maximize lifetime utility subject to the period-by-period budget constraints

$$\begin{aligned} c_1 + s &= w_1 \\ c_2 &= s + w_2, \end{aligned}$$

and the borrowing constraint, $s \geq 0$ (traveling forth and back in time to grab some of the future income w_2 for consumption today is ruled out). Thus, the equilibrium looks as if the agent would face a net return on savings equal to zero and a cost of borrowing that is prohibitively high. So, the equilibrium savings is either positive, or at the corner solution with $s = 0$. Once capital is introduced into the model, agents prefer to save in productive capital instead of just storing consumption as the return on doings so is strictly greater, $r > 0$. Moreover, savings will be strictly positive as, the interest rate is huge when savings and therefore physical capital are close to zero. That provides the necessary incentives for the agents to save.

Problem Set 4

Q: *Exercise 4.1 (a): Why does the Ricardian equivalence proposition not apply to this economy? How could it be restored?*

A: I forgot to reason this in one of the seminars, so I would like to repeat it here and discuss also the matter of intergenerational altruism which could restore the equivalence.

The Ricardian equivalence does not apply because each generation has a finite life (each generation lives for only one period), so the preferred policy of each generation would be to get a big transfer T_t in combination with the accumulation of a lot of government debt. The reason is that the next generation and not the current one has to pay back the government debt. So, the essential part is that the current generation does not care about the next generation.

Suppose instead that each generation would care about the next one (perfect intergenerational altruism) to the extent that it would see the accumulation of government debt as an equivalent tax liability for the future generation, then the Ricardian equivalence proposition would be restored. Such an economy with perfect intergenerational altruism can be shown to be observationally equivalent to an economy with an infinitely-lived household.

Problem Set 6

Q: *Exercise 6.1 (a): Could I also use the probability weighted terms λ_0, $\beta p \lambda_1(s_G)$, and $\beta(1-p)\lambda_1(s_B)$ as the Lagrange multipliers on the state-by-state constraints?*

A: Yes, you are pretty free in choosing the form of the Lagrange multipliers (but remember that this changes the interpretation of the $\lambda_1(s_1)$ a bit). It tuns out that the suggested form is particularly convenient to write the Lagrangian in compact form

$$\begin{aligned}\mathcal{L} = u(c_0) - v(h_0) + \lambda_0 \left[w_0 h_0 - c_0 - a_1\right] \\ + \beta \mathrm{E}\Big[u(c_1(s_1)) - v(h_1(s_1)) \\ + \lambda_1(s_1) \left[w(s_1)h_1(s_1) + (1+r_1)a_1 - c_1(s_1))\right]\Big].\end{aligned}$$

Q: *Exercise 6.1 (d): Can you put straight the confusion about Jensen's inequality?*

A: Let's see. Here's a try:

(1) A twice differentiable function $g(x)$ is (strictly) convex on (a, b) if and only if for each $x \in (a, b)$ we have $g''(x)(>) \geq 0$.

(2) A twice differentiable function $h(x)$ is (strictly) concave on (a, b) if and only if $-h(x)$ is (strictly) convex.

(3) **Jensen's inequality (convex function):**
Let $g(x)$ be a convex function on (a, b), and x a random variable, then

$$g(\mathrm{E}[x]) \leq \mathrm{E}[g(x)].$$

(3b) Remark: If $g(x)$ is strictly convex, then equality in Jensen's inequality occurs only if x is deterministic.

(4) **Jensen's inequality (concave function):**
Let $h(x)$ be a concave function, such that $-h(x)$ is a convex function. Applying Jensen's inequality for a convex function above yields

$$-h(\mathrm{E}[x]) \leq \mathrm{E}[-h(x)] = -\mathrm{E}[h(x)].$$

Multiply both sides of the inequality by (-1), such that also the relation flips

$$h(\mathrm{E}[x]) \geq \mathrm{E}[h(x)].$$

This is Jensen's inequality for concave functions.

(4b) If $h(x)$ is strictly concave, then equality in the above inequality occurs only if x is deterministic.

(5) Most of the utility functions we worked with in the seminar are strictly concave, and have a strictly convex marginal utility (log-utility for example). Thus, you have to apply the appropriate Jensen inequality to (strictly concave) utility, and (strictly convex) marginal utility, respectively.

Q: *Exercise 6.1 (f): Here is the correct derivation of the implicit function's derivative that I messed up in the last 8-10 Monday seminar.*

A: Here is it, you could also stop at the third line, as you can see from the expression that it must be strictly positive as long as $\varphi > 0$:

$$\begin{aligned}
\frac{dh_0(\tilde{a}_1)}{d\tilde{a}_1} &= -\frac{\partial G(h_0(\tilde{a}_1), \tilde{a}_1)/\partial \tilde{a}_1}{\partial G(h_0(\tilde{a}_1), \tilde{a}_1)/\partial h_0(\tilde{a}_1)} \\
&= \frac{w_0^{\varphi}(-\varphi)(w_0 h_0(\tilde{a}_1) - \tilde{a}_1)^{-\varphi-1}(-1)}{w_0^{\varphi}(-\varphi)(w_0 h_0(\tilde{a}_1) - \tilde{a}_1)^{-\varphi-1} w_0 - 1} \\
&= \frac{\varphi w_0^{\varphi}(w_0 h_0(\tilde{a}_1) - \tilde{a}_1)^{-\varphi-1}}{\varphi w_0^{\varphi}(w_0 h_0(\tilde{a}_1) - \tilde{a}_1)^{-\varphi-1} w_0 + 1} \\
&= \frac{\varphi w_0^{\varphi}(w_0 h_0(\tilde{a}_1) - \tilde{a}_1)^{-\varphi}}{\varphi w_0^{\varphi}(w_0 h_0(\tilde{a}_1) - \tilde{a}_1)^{-\varphi} w_0 + (w_0 h_0(\tilde{a}_1) - \tilde{a}_1)} \\
&= \frac{\varphi w_0^{\varphi} h_0(\tilde{a}_1)}{\varphi h_0(\tilde{a}_1) w_0 + (w_0 h_0(\tilde{a}_1) - \tilde{a}_1)} \\
&= \frac{\varphi h_0(\tilde{a}_1)}{(1+\varphi) h_0(\tilde{a}_1) w_0 - \tilde{a}_1} > 0.
\end{aligned}$$

such that household will increase the labor supply in the first period with the level of savings.

Problem Set 7

Q: *A.1: Does the absence of the precautionary savings motive imply that there are no savings?*

A: No, in the seminars we have just cooked up the deterministic equilibrium such that agent's hold zero assets, which is convenient for the computations. We could also have cooked up an equilibrium where the asset holdings are positive in the deterministic economy. What the precautionary savings motive relates to is the additional savings - compared to the deterministic benchmark - that agent's save when the risk in the economy increases (the shift of the asset demand curve). The quadratic utility function in exercise A.1 for example, does not yield a shift of the asset demand as the risk increases.