Problem Set 1: Ramsey's Growth Model

Exercise 1.1: An infinite horizon problem with perfect foresight

In this question we will look at a discrete-time version of Ramsey's growth model. The economy is closed and we consider a representative agent with the following preferences over consumption

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where c_t denotes period t consumption and $\beta \in (0, 1)$ is the subjective discount factor. The momentary utility function is of the form

$$u(c_t) = \frac{c_t^{1-\theta} - 1}{1 - \theta},$$

with $\theta > 1$. Every period the agent earns a wage w_t (the labor supply is exogenously set to 1 unit), interest $r_t a_t$ from her assets and she is subject to a lump-sum tax τ_t. In equilibrium, the agent will choose the sequence consumption and asset holdings $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize U subject to the period-by-period budget constraint

$$c_t + a_{t+1} = w_t + (1 + r_t)a_t - \tau_t, \tag{2}$$

for a given a_0. The agent is atomic and her decisions do not influence aggregate variables, thus she takes the sequence of taxes, wage rates and interest rates as given.

(a) Formulate the Lagrangian of the agent's decision problem (it is common to use λ_t as the Lagrange multiplier on the period t budget constraint). Derive the first-order conditions for the optimal choice of c_t and a_{t+1}, combine these to solve for the consumption Euler equation, and give an (micro theory) interpretation of this equation.

(b) Use the Euler equation to show that the functional form of $u(c_t)$ implies a constant elasticity of intertemporal substitution (EIS) between current and future consumption, where

$$\text{EIS} \equiv \frac{\partial \log(c_{t+1}/c_t)}{\partial \log(1 + r_{t+1})}.$$

Give an (consumption growth) interpretation of the EIS.

The representative firm demands physical capital k_t and labor n_t to produce output y_t with the Cobb-Douglas technology

$$y_t = k_t^{\alpha} n_t^{1-\alpha}. \tag{3}$$

The firm is atomic and acts as a price-taking profit maximizer. Capital can be rented at the rate r_t while labor costs w_t.

(c) Find the first-order conditions for the firm's optimization problem.

The government can raise lump-sum taxes τ_t and rolls over debt in the form of one-period bonds, D_{t+1}, to finance government expenditure, G_t. As it pays an interest rate r_t (we will see later that the depreciation rate δ is indeed zero) on the outstanding debt, D_t, the government faces a period-by-period budget constraint

$$G_t = \tau_t + D_{t+1} - (1 + r_t)D_t. \tag{4}$$

Moreover, assume that the time path of government debt is such that it is growing at a lower rate than the interest rate

$$\lim_{T\to\infty} \frac{D_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0.$$

In other words, it is not feasible for the government to finance the outstanding debt (plus interest payments) by issuing ever more debt as time goes by.

(d) Use the government's budget constraint in Equation (4) and subsitute for d_t iteratively ($t = 1, 2, 3, ..., T$) to derive the government's intertemporal budget constraint in net present value (NPV) terms

$$D_0 = \sum_{t=0}^{\infty} \frac{\tau_t - G_t}{\Pi_{s=0}^{t}(1 + r_s)}. \tag{5}$$

Give an interpretation of Equation (5).

(e) Repeat the same exercise for the representative agent's budget constraint in Equation (2) and derive the intertemporal budget constraint in NPV terms. What would be the (trivial) solution to the agent's maximization problem if the no-Ponzi condition

$$\lim_{T\to\infty} \frac{a_{T+1}}{\Pi_{s=0}^{T}(1 + r_s)} = 0$$

was not imposed and assuming that $r_s = r < \infty$?

(f) Assume that Equation (5) holds for a given stream $\{\tau_t, G_t\}_{t=0}^{\infty}$, and so do the no-Ponzi conditions. Consider an increase in government expenditures ΔG_t that can be either financed by raising taxes, τ_t, or government debt, D_{t+1}. Does the agent respond differently to a tax-financed relative to a debt financed increase in government expenditures, if she anticipates the government intertemporal budget constraint? How does your result relate to the Ricardian equivalence proposition?

Remember that the model under consideration is a closed economy and has three markets: the market for labor, the market for consumption goods, and the capital market.

(g) State the three market clearing conditions. Then, solve for the competitive equilibrium variables $\{c_{t+1}, a_{t+1}, k_t, n_t, r_t, w_t, y_t\}_{t=0}^{\infty}$ and the sequence of debt $\{D_{t+1}\}_{t=0}^{\infty}$ as a function of initial consumption c_0, inital assets a_0, inital debt D_0, and the sequence of exogenous government policy $\{G_t, \tau_t\}_{t=0}^{\infty}$ using the first-order conditions, budget constraints and market clearing conditions.

The first welfare theorem applies to this economy such that the competitive equilibrium is efficient in the Pareto sense. Thus, we know that the solution to the social planner's problem (which characterizes the Pareto efficient allocation) is equivalent to the competitive market equilibrium.

(h) Show that the solution to the Bellman equation

$$V(k_t) = \max_{k_{t+1}} u(k_t^\alpha + k_t - G_t - k_{t+1}) + \beta V(k_{t+1}), \tag{6}$$

is the solution to the social planner's problem

$$V(k_t) = \max_{\{c_s, k_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s), \quad \text{s.t.} \quad c_s = k_s^\alpha + k_s - G_s - k_{s+1},$$

for a given $k_0 > 0$. Note that instead of imposing a no-Ponzi condition, we will implicitly require throughout that physical capital and consumption remain positive, $k_{t+1} \geq 0$, $c_t \geq 0$.

According to the social planner's solution the consumption Euler equation and the resource constraint

$$\frac{c_{t+1}}{c_t} = [\beta(1 + r_{t+1})]^{1/\theta} = \left[\beta(1 + \alpha k_{t+1}^{\alpha-1})\right]^{1/\theta}$$
$$k_{t+1} - k_t = k_t^\alpha - c_t - G_t.$$

determine the dynamics of the system. Let us assume that $G_t = G$, then we can define two correspondances. One which characterizes all possible combinations of (c_t, k_t) when consumption is constant,

$$\mathcal{C}_1(k) \equiv \left\{ c \in [0, \infty) : c_{t+1}/c_t = \left[\beta(1 + \alpha k^{\alpha-1})\right]^{1/\theta}, c_{t+1} = c_t = c \right\},$$

and one which captures all combinations when the physical capital stock is constant,

$$\mathcal{C}_2(k) \equiv \{ c \in [0, \infty) : c = k_t^a - (k_{t+1} - k_t) - G, k_{t+1} = k_t = k \}.$$

(i) Draw the two correspondances, $\mathcal{C}_1(k)$ and $\mathcal{C}_2(k)$, in a diagram with k on the horizontal axis and c on the vertical axis, the so called phase diagram.

(j) Comment on the unique point in the phase diagram where the two correspondances intersect.

(k) Using the phase diagram, illustrate in what direction (c_t, k_t) will move (in all areas of the (c, k)-space).

(l) Sketch (we do not know the precise shape at this stage) the saddle path leading to the steady state. Explain why any initial consumption off the saddle path cannot be an equilibrium.

(m) Consider the steady state consumption level as a function of physical capital

$$c = k^{\alpha} - G.$$

The Golden Rule capital stock is defined as the physical capital stock that maximizes steady-state consumption. Compare the steady state real interest rate of the Ramsey model with the real interest rate that would prevail under the Golden Rule. Is the steady-state physical capital stock in the Ramsey model lower or higher than under the golden rule? Why is that so?