Problem Set 2:
More on Ramsey's Growth Model

Exercise 2.1: Public spending in Ramsey's growth model

Reconsider the model setup of Exercise 1.1 and remind yourself of the phase diagram we derived there. Assume that the economy is in a stationary equilibrium with constant government expenditures, $G_t = G$, and tax policy, $\tau_t = \tau$.

(a) Consider an unexpected and temporary cut of ΔG in government expenditures from period t_0 until period t_1. Sketch the dynamics of consumption and physical capital back to the steady-state in the phase diagram.

(b) Consider an unexpected and permanent cut of ΔG in government expenditures for all future periods $t \geq t_0$. Sketch the dynamics of consumption and physical capital to the new steady-state in the phase diagram.

(c) Sketch the time path of the wage rate for both of the scenarios described in Exercise 2.1 (a) and (b).

(d) Consider instead an unexpected and temporary decrease of $\Delta\beta$ in the discount factor from period t_0 to period t_1. Sketch the dynamics of consumption and physical capital back to the steady-state in the phase diagram.

Exercise 2.2: Ramsey model in discrete time: a closed form solution

Consider the following version of the Ramsey model with exogenous technology and population growth

$$\max U = \sum_{t=0}^{\infty} \beta^t \log(c_t A_t) L_t, \quad 0 < \beta < 1,$$

subject to

$$\begin{aligned} K_{t+1} &= K_t^{\alpha}(A_t L_t)^{1-\alpha} + (1-\delta)K_t - C_t, \quad 0 < \alpha < 1, \\ A_{t+1} &= (1+g)A_t, \quad A_0 > 0, g > 0, \\ L_{t+1} &= (1+n)L_t, \quad L_0 > 0, n > 0, \end{aligned}$$

where we require that c_t and K_{t+1} remain non-negative, and $K_0 > 0$ is taken as given. The parameter δ denotes the depreciation rate, C_t is aggregate consumption, K_t aggregate physical capital, L_t aggregate labor supply, Y_t aggregate production, A_t labor augmenting productivity, and lower case variables correspond the the same variable in efficiency units $x_t \equiv X_t/(A_t L_t)$.

(a) Remove the trend growth from the capital accumulation equation by restating it in terms of consumption and physical capital per efficiency unit, $c_t \equiv C_t/(A_t L_t)$ and $k_t \equiv K_t/(A_t L_t)$, respectively.

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(b) State the Lagrangian and derive the first-order conditions with respect to consumption c_t and physical capital k_{t+1} per efficiency unit.

(c) Set the depreciation rate of physical capital to $\delta = 1$, and derive the consumption Euler equation. Guess that

$$c_t(k_t) = \gamma k_t^{\alpha}$$

is the solution to the first-order conditions derived above (this corresponds to the stable saddle-path of the economy). What must be the value of the constant γ in equilibrium? (hint: plug the guess into the Euler equation to derive $k_{t+1}(k_t)$ and then use the capital accumulation equation to determine the constant γ.)

(d) What is the steady-state physical capital stock per efficiency unit, $k^{\star}$, in this economy? Sketch the associated phase diagram including the saddle-path with its correct shape.

Exercise 2.3: A finite horizon problem with perfect foresight

Consider the following problem of intertemporal consumption choice in discrete and finite time, $t = 0, 1, \ldots, T < \infty$,

$$\max \sum_{t=0}^{T} \beta^t u(c_t), \quad \text{s.t.} \quad a_{t+1} = (1 + r_t)a_t + w_t - c_t, \ \forall t,$$

with $a_0 > 0$, $\{r_t\}_{t=0}^{T}$, $\{w_t\}_{t=0}^{T}$ given, and $0 < \beta < 1$ (the notation is the same as in Exercise 1.1). The trivial solution to this problem is to set the terminal debt to infinity, $a_{T+1} = -\infty$, and enjoying infinite utility. To rule out such a solution it is necessary to impose the additional constraint

$$a_{T+1} \geq 0, \tag{1}$$

which means that the agent cannot leave debt at the end of her life. Note that the terminal condition stated in Equation (1) is the analog of the no-Ponzi condition in the corresponding infinite horizon problem.

(a) State the Lagrangian of this optimization problem (hint: use $\beta^t \lambda_t$ as the Lagrange multiplier on the period-by-period budget constraint and treat the constraint in Equation (1) as a regular equality constraint with Lagrange multiplier μ).

(b) State the first-order optimality conditions with respect to c_t, a_{t+1} (for $t < T$), and a_{T+1}. Then complement it with the so called complementary slackness condition (from the Karush-Kuhn-Tucker conditions)

$$\mu a_{T+1} = 0, \ \mu \geq 0.$$

This condition simply means that either $\mu = 0$ and $a_{T+1} \geq 0$ (the inequality constraint is not binding) or $\mu > 0$ and $a_{T+1} = 0$ (the inequality constraint is binding).

(c) Given that marginal utility is positive, $u'(c) > 0$, show that $\mu > 0$ implying that $a_{T+1} = 0$. Next, show that as $T \to \infty$

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0.$$

Comment on this equilibrium condition.

(d) Consider the following preference specification

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \theta > 1.$$

Show that $\lim_{\theta\to 1} u(c) = \log(c)$ (hint: make use of Hôpital's rule).

(e) Firstly, derive the consumption Euler equation of this dynamic system. Then set the parameters of the model to $\beta = 97/100$, $T = 44$, $\theta = 2$, $a_0 = 0$, $r_t = 4/100$, $w_t = 1/2$, and use the Euler equation as well as the period-by-period budget constraint to solve for the optimal consumption and asset accumlation paths. You will have to do the calculations with your preferred spreadsheet application (hint: guess a c_0 and then solve the dynamic system forward. If $a_{T+1} > 0$, then you need to increase the inital guess c_0 as leaving assets on the table at the end of your life instead of consuming them cannot be optimal. On the other hand, if $a_{T+1} < 0$, then you need to lower c_0. You may want to use a solver which is available in most spreadsheet applications to solve for the optimal value c_0 such that $a_{T+1} = 0$ in one step).

(f) Using again your spreadsheet application, plot the optimal consumption and asset accumulation path in a diagram with t on the horizontal axis. Does the agent hold negative assets in any of the periods, and what parameter constellation would change this pattern?

(g) Suppose that $w_{30} = 0$ instead of $w_{30} = 1/2$. How do the optimal consumption and asset accumulation paths change?