Problem Set 2:
More on Ramsey's Growth Model (Partial Solution)

Exercise 2.3: A finite horizon problem with perfect foresight

Consider the following problem of intertemporal consumption choice in discrete and finite time, $t = 0, 1, \ldots, T < \infty$,

$$\max \sum_{t=0}^{T} \beta^t u(c_t), \quad \text{s.t.} \quad a_{t+1} = (1 + r_t)a_t + w_t - c_t, \, \forall t,$$

with $a_0 > 0$, $\{r_t\}_{t=0}^{T}$, $\{w_t\}_{t=0}^{T}$ given, and $0 < \beta < 1$ (the notation is the same as in Exercise 1.1). The trivial solution to this problem is to set the terminal debt to infinity, $a_{T+1} = -\infty$, and enjoying infinite utility. To rule out such a solution it is necessary to impose the additional constraint

$$a_{T+1} \geq 0, \tag{1}$$

which means that the agent cannot leave debt at the end of her life. Note that the terminal condition stated in Equation (1) is the analog of the no-Ponzi condition in the corresponding infinite horizon problem.

(a) State the Lagrangian of this optimization problem (hint: use $\beta^t \lambda_t$ as the Lagrange multiplier on the period-by-period budget constraint and treat the constraint in Equation (1) as a regular equality constraint with Lagrange multiplier μ).

(b) State the first-order optimality conditions with respect to c_t, a_{t+1} (for $t < T$), and a_{T+1}. Then complement it with the so called complementary slackness condition (from the Karush-Kuhn-Tucker conditions)

$$\mu a_{T+1} = 0, \, \mu \geq 0.$$

This condition simply means that either $\mu = 0$ and $a_{T+1} \geq 0$ (the inequality constraint is not binding) or $\mu > 0$ and $a_{T+1} = 0$ (the inequality constraint is binding).

(c) Given that marginal utility is positive, $u'(c) > 0$, show that $\mu > 0$ implying that $a_{T+1} = 0$. Next, show that as $T \to \infty$

$$\lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0.$$

Comment on this equilibrium condition.

(d) Consider the following preference specification

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta}, \, \theta > 1.$$

Show that $\lim_{\theta \to 1} u(c) = \log(c)$ (hint: make use of Hôpital's rule).

(e) Firstly, derive the consumption Euler equation of this dynamic system. Then set the parameters of the model to $\beta = 97/100$, $T = 44$, $\theta = 2$, $a_0 = 0$, $r_t = 4/100$, $w_t = 1/2$, and use the Euler equation as well as the period-by-period budget constraint to solve for the optimal consumption and asset accumulation paths. You will have to do the calculations with your preferred spreadsheet application (hint: guess a c_0 and then solve the dynamic system forward. If $a_{T+1} > 0$, then you need to increase the inital guess c_0 as leaving assets on the table at the end of your life instead of consuming them cannot be optimal. On the other hand, if $a_{T+1} < 0$, then you need to lower c_0. You may want to use a solver which is available in most spreadsheet applications to solve for the optimal value c_0 such that $a_{T+1} = 0$ in one step).

Solution:

The consumption Euler equation can be derived from the first-order conditions by eliminating λ_t and using the specification of $u(c)$

$$\frac{c_{t+1}}{c_t} = \left[\beta(1 + r_{t+1})\right]^{1/\theta} . \tag{2}$$

Thus, given a guess for c_0 Equation (2) yields c_1, while the period-by-period budget constraint delivers a_1. You can repeat this procedure sequentially, until you arrive at a_{T+1}. The numerical solution reported in the MS Excel file OptimalConsumption.xlsx yields the optimal initial consumption level of $c_0 = 0.466$.

(f) Using again your spreadsheet application, plot the optimal consumption and asset accumulation path in a diagram with t on the horizontal axis. Does the agent hold negative assets in any of the periods, and what parameter constellation would change this pattern?

Solution:

The agent holds in any period nonnegative assets levels. The reason for this is that $\beta(1 + r) > 1$ which implies that consumption growth is positive, or the agent shifts consumption to future periods by accumulating assets in the early periods. Changing parameters such that $\beta(1 + r) < 1$ reverses this pattern. The agent will shift consumption from future to earlier periods by accumulation debt in the early periods.

(g) Suppose that $w_{30} = 0$ instead of $w_{30} = 1/2$. How do the optimal consumption and asset accumulation paths change?

Solution:

The optimal consumption path remains smooth in period 30 as the consumption Euler equation is unaffected by the drop in wage income. However, the optimal consumption level will be slightly lower in all periods as the lifetime wage income drops. The effect on asset accumulation on the other hand is more pronounced: the agent accumulates more assets until period 30, and as the wage income drops the asset level is immediately adjusted downwards to keep consumption smooth.